PROCEEDINGS

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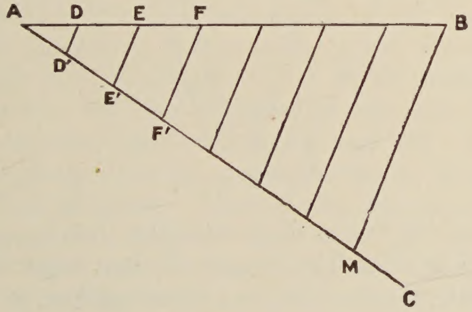
I. *The Uses of a Line-Divider.*

By Miss SARAH MARKS*

THE ordinary method of dividing a given straight line AB into any number of equal parts (fig. 1), without using a special instrument, is to draw a line AC at any angle to it, to

Fig. 1.



cut off from AC the given number of equal parts AD′, D′E′, &c., to join M, the last of the points so taken, to B, and from D′, E′, &c. to draw lines parallel to MB, meeting AB in D, E, &c. These points D, E, &c. divide AB into the required number

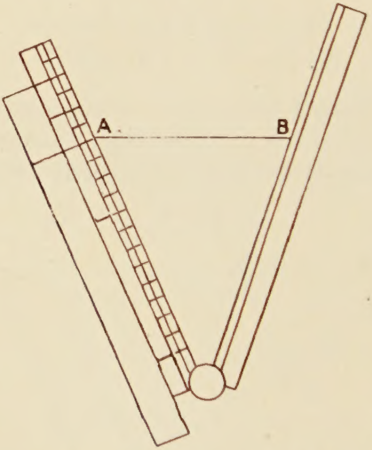
* Read February 14, 1885

of equal parts. This is exactly the method adopted in using the Line-divider.

It consists of a hinged rule with a firm joint, having the left-hand limb fitted to slide in an undercut groove upon the plain rule. Both limbs are bevelled on their inner edges, and the left one is divided both on the bevel and on the top into eighths, quarters, half-inches, and inches, which are consecutively numbered, beginning at the hinged end, so that any set may be used. The plain rule has two needle-points on its underside to prevent it from slipping when placed in any position, and on the top has a single line drawn perpendicular to its longer edges.

To divide the line AB into any number of equal parts, say seven (fig. 2): slide the plain rule along the divided limb till its line coincides with one of the lines of this limb against which the number 7 is marked (it is generally best to use the largest possible divisions), place the corresponding line on the bevel of the limb on the point A, and open or close the rule till the bevel of the undivided limb is on the point B. Now press the plain rule down so that the needle-points enter the paper, and keep it in position. Slide the hinged rule up till the line numbered 6 on the line of reference is coincident with the line on the plain rule, and mark the point where the bevel of the undivided limb meets the line AB. Continue to move the hinged rule up one division at a time till the whole line is

Fig. 2.



divided. It is evident that the ordinary method has been used; for the divided limb forms a line at some angle to AB, and is divided into seven equal parts, and against the undivided limb I can draw parallel lines through the points of section of the divided limb cutting AB in points equidistant from one another.

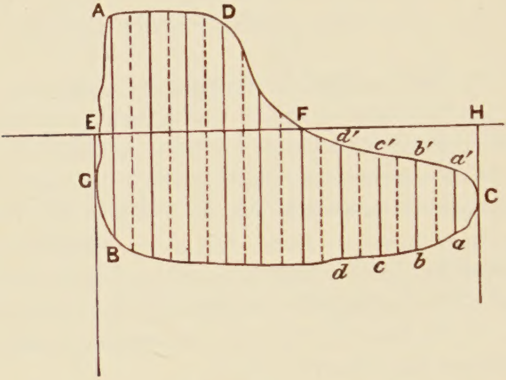
This is the first method of using the instrument, and is useful in many ways :—to artists, for squaring out their canvases when they wish to enlarge or diminish their drawings, although the second method would perhaps be best for this if a very large divider were used ; to decorators, who frequently find it necessary to divide lines into given numbers of equal parts in order to get patterns into a certain space; it would also be useful for finding the divisions on the scales of thermometers, as the distance between the freezing- and boiling-points is always an unknown length.

The second method of using the instrument is this :—Let ABCD be an area to be divided into any number of parts, say five, by equidistant lines parallel to AD or BC. Produce AD both ways : place the bevelled edge of the undivided limb along BC, open or close the rule till the end of one of the lines marked 5 on the bevel of the divided limb is on the line AD. Slide the plain rule up, taking care that it is held high enough for its points not to tear the paper, till its line coincides with the 5-line, and press the points firmly into the paper. Now slide the hinged rule up till the division marked 4 coincides with the line on the plain rule, and draw a line along the undivided limb cutting AB and CD. Bring each of the other three divisions against the line on the plain rule, and draw lines as before. These are the lines required.

It is evident that, since the angle ABC may be any angle, it may be a right angle, so that lines can be drawn perpendicular to AB, dividing it into any given number of equal parts. This method is used in finding the mean pressure in gas- and steam-engines by means of an indicator-diagram.

Let A B C D (fig. 3) be an indicator-diagram, E F the atmospheric line ; G E, H C, tangents at the extremities of the diagram perpendicular to E F. To find the length of the line which represents the mean pressure of the engine, divide the area E H C G into ten parts by lines parallel to H C and E B, and equidistant from one another; bisect each of the areas thus obtained by lines parallel to the last. Let the points where these last lines meet the diagram be *a a′*, *b b′*, &c. . . . .

Fig. 3.



Add together the lengths *a a′*, *b b′*, &c., and divide by 10; the result will give the line representing the mean pressure of the engine. This process of dividing the diagram, first by ten equidistant lines and then bisecting the spaces between these lines, is generally a very troublesome one, effected by means of a system of parallel rulers and a T-square. In the figure the dotted lines are those not required when the line-divider is used.

To divide the diagram by means of the rule, place the rule as if for dividing the area by 20 equidistant lines parallel to HC. Draw the first of these lines, and then every alternate one, and the diagram is divided as required.

This method of dividing areas is also convenient for drawing treads and rises of stairs, joists, roof-timbers.

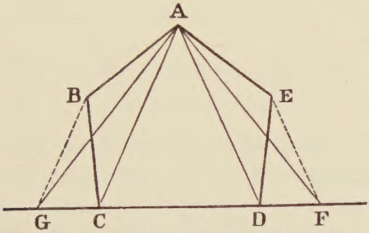
In the third method the divider is used simply as a parallel ruler. As such it has the advantage of having a wide range, and of being firmly fixed at the same time. In graphical statics, where it is often necessary to draw a line parallel to another at a considerable distance from it, set squares have often to be placed and replaced five or six times before the necessary line can be drawn. This instrument will at once draw the line. The range as a parallel ruler may be still further increased by having the plain rule considerably longer than the hinged one.

The special advantages of this instrument as a parallel ruler may best be seen by drawing what surveyors call the "give and take line."

When two fields are divided by a curved boundary, it is often required to find a straight boundary which will divide the fields in the same proportions as before. This straight boundary is what is called the "give and take" line.

In order to understand the method of finding it, it will be necessary to recall the method of finding a rectangle which shall be equal to a given polygon.

Fig. 4.



Let A B C D E (fig. 4) be the given polygon: produce C D both ways, and join A D and A C.

Draw EF ‖ to AD, and BG ‖ to AC, and join AG, AF. Then △ A F G is equal in area to the polygon A B C D E. For ∵ △'s A F D and A E D are on the same base A D and between the same ‖'s A D, E F,

∴ they are equal.

Similarly △ A G C = △ A B C.

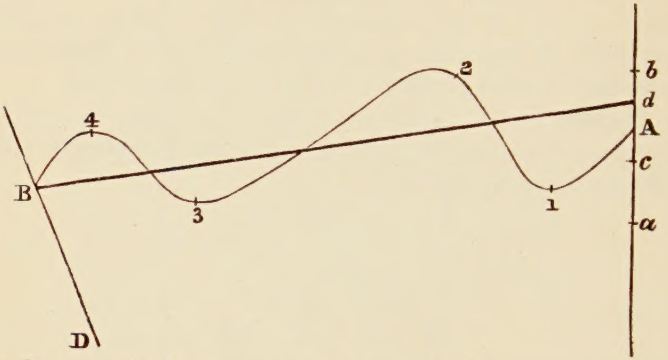
Having this triangle, it is easy to draw a rectangle equal to it.

Now let A 1 2 3 4 B (fig. 5) be the curved boundary of a field of which AC, BD are fixed boundaries; it is required to find a straight boundary to replace the curved one through A and B.

Place the divided limb roughly parallel to AC, with its bevelled edge nearest to AC, and at a distance about equal to the breadth of either limb from it, and fix the points firmly in the paper. Now move the undivided limb till it cuts AC at A and 1 2 at 2; slide it along till it is on the point 1, and mark the point *a* where it meets AC. Move it till it is against *a* and 3, and slide it along till it is against 2, marking the point *b* where it meets AC produced, if necessary.

Move it till it is on *b* and 4, slide it till it is on 3, and mark *c*, the point where it meets AC. Move it on to *c* B, slide back to 4, and mark point *d* where it meets AC; join *d* B. *d* B is the line required, as may easily be seen by comparing the portions given and taken by the respective fields. This line would only divide the fields perfectly accurately as before, if A 1 2, 2 3 4, &c., were rectilineal angles ; but it is the nearest approximation possible, and the one always used.

Fig. 5.



The divider would also be useful on board ships for drawing lines on charts parallel to other lines at some distance. It would be much better for this purpose than an ordinary parallel ruler on account of its very large range and capability of being fixed.

Of course the line-divider can be used in all cases in which parallel rulers are ordinarily used.

In summing up, the advantages of the line-divider appear to be these :—

1. Lines of division can be drawn at any angle to the line to be divided.

2. Areas can be divided by equidistant lines parallel to any given line.

3. Since the divisions on the instrument are all equal, they may conveniently be made inches and parts of an inch ; and

4, consequently the number of divisions may be made as great as we like without materially increasing the cost.

5. Lastly, the cost of the instrument is small, and it is made by Messrs. Stanley in three sizes—6, 12, and 24 inches in length.

II. *On an Integrating Anemometer.*

By Walter Baily, *M.A.*[*]

To this anemometer, which had been already described (Phys. Soc. Proc. vol. vi. p. 115), mechanical counters similar to those used for bicycles had been added, as it has been found that the electric counters with which the instrument was originally made were liable to let the batteries run down when the wind dropped, and were therefore only suited for observations of short duration.

III. *On the Quadrant-Electrometer.*

By J. Hopkinson, *D.Sc., F.R.S.*[†]

In Professor Clerk Maxwell's 'Electricity' (vol. i. p. 273) it is proved that the deflection of the needle of a quadrant-electrometer varies as $(A-B)\left(C-\frac{A+B}{2}\right)$, where C is the potential of the needle, and A and B of the two pairs of quadrants. Desiring to ascertain the value of the standard charge of my instrument, I endeavoured to do so by the aid of this formula, and also by a more direct method. The results were quite discordant. Setting aside the special reasoning by which the formula is attained, we should confidently expect that the sensibility of a quadrant-electrometer would increase continuously as the charge of the jar is increased, until at last a disruptive discharge occurs. In my instrument this is not the fact. As the charge was steadily increased by means of the replenisher, the deflection of the needle due to three Daniell's elements at first increased, then attained a maximum, and with further increase of charge actually diminished. On turning the replenisher in the inverse direction the sensibility at first increased, attained the maximum previously observed, and only on further reduction of charge diminished.

Before giving the experimental results, it may be worth while to briefly examine the theory of the quadrant-electrometer. Let A, B, C, D be the potentials of the quadrants, the needle, and the inductor which is used for measuring high potentials (see Reprint of Sir W. Thomson's papers, p. 278). Let Q_1, Q_2, Q_3, Q_4 be the quantities of electricity on these bodies respectively, and θ the angle of deflection of the needle, measured in terms of divisions of the scale, on which the image of the lamp-flame is projected. We have the equations

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[*] Read February 14, 1885.

[†] Read March 14, 1885.

$$\left.\begin{aligned} Q_1 &= \quad q_{11}A - q_{12}B - q_{13}C - q_{14}D, \\ Q_2 &= -q_{12}A + q_{22}B - q_{23}C - q_{24}D, \\ Q_3 &= -q_{13}A - q_{23}B + q_{33}C - q_{34}D, \\ Q_4 &= -q_{14}A - q_{24}B - q_{34}C + q_{44}D. \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

q_{11} &c. are the coefficients of capacity and induction. They are independent of A, B, C, D, and are functions of θ only. As above written, they are all positive. Let the energy of electrification be W:—

$$\left.\begin{aligned} 2W = \; & q_{11}A^2 + q_{22}B^2 + q_{33}C^2 + q_{44}D^2 \\ & -2q_{12}AB - 2q_{13}AC - 2q_{14}AD \\ & -2q_{23}BC - 2q_{24}BD \\ & -2q_{34}CD \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

Equations (1) and (2) are perfectly general, true whatever be the form of the four bodies.

If the four quadrants completely surround the needle,

$$\left.\begin{aligned} & q_{34} = 0, \\ & q_{14}, q_{24}, \text{ and } q_{44} \text{ are independent of } \theta, \\ & q_{33} = q_{13} + q_{23}. \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

Now when the electrometer is properly adjusted, the needle will not be deflected when $A=B$, whatever C and A may be. Hence $A-B$ is a factor of $\frac{dW}{d\theta}$, and we have

$$\left.\begin{aligned} & \frac{dq_{11}}{d\theta} + \frac{dq_{22}}{d\theta} = 2\frac{dq_{12}}{d\theta}, \\ & \frac{dq_{33}}{d\theta} = 0 ; \end{aligned}\right\}$$

whence

$$2\frac{dW}{d\theta} = (A-B)\left(A\frac{dq_{11}}{d\theta} - B\frac{dq_{22}}{d\theta} - C\frac{dq_{13}}{d\theta}\right).$$

This should be true of any electrometer having the above adjustment correctly made.

But by suitably forming the three bodies A, B, C, further relations between the coefficients may be obtained. The condition of symmetry would give us $\frac{dq_{11}}{d\theta} = -\frac{dq_{22}}{d\theta}$; but it is not necessary to assume symmetry. If the circumferential termination of the needle be a circle centre in the axis of suspension (at least near the division of the quadrants), if the needle turn in its own plane, if the quadrants are each approximately a surface of revolution about the axis, and if the radial terminations of the needle be not within the electrical influence of the quadrants within which they are not, conditions closely satisfied in Sir W. Thomson's electrometer,

$$\frac{dq_{11}}{d\theta} = \frac{1}{2}\frac{dq_{13}}{d\theta},$$

$$\frac{dq_{22}}{d\theta} = -\frac{1}{2}\frac{dq_{13}}{d\theta}.$$

If θ be small, we obtain

$$2\frac{dW}{d\theta} = \alpha(A-B)\left(C-\frac{A+B}{2}\right),$$

the formula in Maxwell.

Returning now to our original equation, we have

$$\begin{aligned}2W = {} & q_{11}A^2 + q_{22}B^2 + q_{33}C^2 + q_{44}D^2 \\ & -2q_{12}AB - q_{33}AC - 2q_{14}AD \\ & -\ q_{33}BC - 2q_{24}BD \\ & + \alpha\theta(A-B)\left(C-\frac{A+B}{2}\right);\end{aligned}$$

involving in all eight constants, q_{11} &c. being now regarded as representing the values of the coefficients in the zero position.

$$\left.\begin{aligned}Q_1 &= \quad q_{11}A - \quad q_{12}B - \tfrac{1}{2}q_{33}C - q_{14}D + \alpha\theta(C-A),\\ Q_2 &= -\ q_{12}A + \quad q_{22}B - \tfrac{1}{2}q_{33}C - q_{24}D - \alpha\theta(C-B),\\ Q_3 &= -\tfrac{1}{2}q_{33}A - \tfrac{1}{2}q_{33}B + \quad q_{33}C \qquad\qquad + \alpha\theta(A-B),\\ Q_4 &= -\ q_{14}A - \quad q_{24}B \qquad\qquad + q_{44}D.\end{aligned}\right\}$$

We may now discuss a variety of important particular cases.

(*a*) B is put to earth; A then is connected to a condenser, capacity *a*, charged to potential V: we want to know V from the reading of the electrometer. Here

$$aV-\tfrac{1}{2}q_{33}C=(a+q_{11})A-\tfrac{1}{2}q_{33}C+\alpha\theta(C-A),$$

$$V=A+\frac{q_{11}}{a}A+\frac{\alpha\theta(C-A)}{a}.$$

Neglecting A compared with C, and assuming

$$\theta=\lambda(A-B)\left(C-\frac{A+B}{2}\right),$$

we have

$$V=A\left\{1+\frac{q_{11}}{a}+\frac{\lambda\alpha C^2}{a}\right\}.$$

The apparent capacity of A increases with C.

(*b*) B is again zero. A is connected to a source, but is disconnected and insulated when the deflection of the needle is θ'; the final deflection is θ: required the potential V of the source.

$$\begin{aligned} q_{11}V-\tfrac{1}{2}q_{33}C+\alpha\theta'(C-V)&\\ =q_{11}A-\tfrac{1}{2}q_{33}C+\alpha\theta(C-A),&\\ V=A+\frac{\alpha(\theta-\theta')C}{q_{11}}&\\ =A\left\{1+\frac{\alpha\lambda\left(1-\frac{\theta'}{\theta}\right)C^2}{q_{11}}\right\}.& \end{aligned}$$

We may now consider the methods of varying the sensibility of the instrument (see Reprint of Sir W. Thomson's papers, p. 280). The methods dealt with are those of Sir W. Thomson, somewhat generalized.

(*c*) The quadrant B is connected with an insulated condenser, capacity *b*, whilst A is connected to a source of electricity:—

$$0=-q_{12}A+(b+q_{22})B-\alpha\theta C,$$

$$\theta=\lambda(A-B)C;$$

therefore

$$0=-q_{12}A+(b+q_{22})\left(A-\frac{\theta}{\lambda C}\right)-\alpha\theta C;$$

so

$$A=\theta\frac{\dfrac{b+q_{22}}{\lambda C}+\alpha C}{(b+q_{22})-q_{12}}.$$

If $b=0$, we have the first reduced sensibility given by Sir W. Thomson.

(*d*) All methods of using the inductor may be treated under one general form. Let the quadrants A and B be connected with insulated condensers, capacities a and b; then connect the inductor to a source, potential V;

$$\left.\begin{aligned} 0&=(q_{11}+a)\mathrm{A}-q_{12}\mathrm{B}-q_{14}\mathrm{V}+a\theta\mathrm{C}, \\ 0&=-q_{12}\mathrm{A}+(q_{22}+b)\mathrm{B}-q_{24}\mathrm{V}-a\theta\mathrm{C}, \\ 0&=\lambda(\mathrm{A}-\mathrm{B})\mathrm{C}; \end{aligned}\right\}$$

$$\begin{aligned} (+q_{11}q_{22}-q_{12}^2+aq_{22}+bq_{11}+ab)(\mathrm{A}-\mathrm{B}) \\ +\{q_{14}(q_{12}-q_{22}-b)-q_{24}(q_{12}-q_{11}-a)\}\mathrm{V} \\ +\{-2q_{12}+q_{22}+q_{11}+b+a\}a\theta\mathrm{C}=0\ ; \end{aligned}$$

whence we have an expression for V proportional to θ. By a proper choice of a and b, we can make the sensibility as low as we please.

Now the whole of these formulæ rest on the same reasoning as the equation

$$\theta=\lambda(\mathrm{A}-\mathrm{B})\left(\mathrm{C}-\frac{\mathrm{A}+\mathrm{B}}{2}\right).$$

I have mentioned that, in my instrument at least, this equation quite fails to represent the facts when C is considerable. It becomes a matter of interest to ascertain when the formula begins to err to a sensible extent. If a constant battery of a large number of elements were available, this would be soon accomplished. I have at present set up only 18 Daniells. I have therefore been content to use the electrometer to ascertain its own charge by the aid of the inductor, using the 18 Daniells as a standard potential. As the charges range as high as 2600 Daniell's elements, the higher numbers can only be regarded as very rough approximations; sufficiently near, however, to indicate the sort of result which would be obtained if more precise methods were used. The first column in the following Table gives the ascertained or estimated charge of the jar of my electrometer in Daniell's elements; the second the deflection in scale-divisions caused by three elements; the third, the coefficient λ, deduced by the formula $\theta=\lambda\mathrm{AC}$: this coefficient ought theoretically to be constant.

I.	II.	III.
72	75	0·35
112	118	0·35
136	140	0·35
178	190	0·35
238	239	0·34
303	288	0·32
383	336	0·30
512	391	0·26
616	409	0·22
813	432	0·18
1080	424	0·13
1312	402	0·10
1728	360	0·07
2124	320	0·05
2634	296	0·037
1704	353	0·07
1436	394	0·09
1284	412	0·11
876	436	0·17
684	427	0·21

By connecting the jar and one quadrant to 18 elements and the other quadrant to earth, I obtained 0·356 as the value of λ, making use of the complete equation

$$\theta=\lambda(A-B)\left(C-\frac{A+B}{2}\right).$$

It will be seen that this equation may be trusted until C is over 200 Daniell's elements potential, but that when C exceeds 250 a quite different law rules.

The foregoing was read before the Physical Society a few years ago, but I stopped its publication after the type was set up, because I was not satisfied that my appliances for experiment were satisfactory, or that I could give any satisfactory explanation of the anomaly.

The electrometer had been many times adjusted for various purposes before further experiments were made, so that those which I shall now describe cannot be directly compared with what goes before. The old experiment was first repeated, and the existence of a maximum sensibility again found. On examination, it was found that the needle hung a little low so that it was nearer to the part of the quadrant below it than to that above. It is easy to see that this would produce the anomalous result observed, though there is reason for thinking it is not the sole cause. The effect of the needle being low is that it will be on the whole attracted downwards; and so the apparent weight hanging on the fibre-suspension and the consequent tension of the fibres will be increased. The increase of the tension will be as the square of the potential C; and hence the formula for the deflection will be modified to

$$\theta = \frac{\lambda}{1+k\mathrm{C}^2}(\mathrm{A}-\mathrm{B})\left(\mathrm{C}-\frac{\mathrm{A}+\mathrm{B}}{2}\right),$$

where k is a constant depending upon the extent to which the position of the needle deviates from its true position of midway between the upper and lower parts of the quadrants. By a proper choice of k, the results I previously obtained are found to agree well with this formula.

The electrometer was next adjusted in the following way:—The needle was raised by taking up the fibres of the suspension and adjusting them to equal tension in the usual way, and the proportionality of sensibility to charge was tested, the charge being now determined in arbitrary units by discharging the jar of the instrument through a ballistic galvanometer. The operation was repeated until the sensibility, so far as this method of testing goes, was proportional to the charge of the jar over a very long range. It was then found that the needle was slightly above the median position within the quadrants. Increased tension of the fibres from electrical attraction does not therefore account for the whole of the facts, although it does play the principal part. The sensibility of the instrument being now at least approximately proportional to the charge of the jar, I proceeded to determine accurately the potential of the jar when charged to the standard as indicated by the idiostatic gauge.

In what follows the quadrants, one of which is under the induction-plate, are denoted by B, the others by A. The quadrants B are connected to the case, A are insulated. The jar is connected to the induction-plate, and the reading on the scale noted; the connection is broken, and the induction-plate is connected to the case, and the reading on the scale again noted; the difference is the deflection due to the charge in the jar. It is necessary to read the scale for zero-charge on induction-plate last, because the charging of the induction-plate slightly diminishes the charge of the jar, and considerably displaces the zero-reading by giving an inductive charge to the quadrant A. It is also necessary to begin with the charge of the jar minutely too high, so that after separating the induction-plate from the interior of the jar, the latter shall have exactly the correct charge as indicated by the gauge. The deflection thus obtained was precisely 298½, repeated in many experiments. The *double* deflection given by seventy Daniell cells was 43·6 scale-divisions. By comparison with two Clark's cells, the value of which I know, the potential of the seventy Daniells was found to be 74·2 volts; hence the potential of the jar is 1016 volts, when charged to the potential indicated by the gauge.

The constant λ of the instrument was next determined by the formula

$$\theta=\lambda(A-B)\left(C-\frac{A+B}{2}\right).$$

Four modes of connecting are available for this :—

$$A=C=74{\cdot}2 \text{ volts, } B=0\,;$$
$$B=C=74{\cdot}2 \text{ volts, } A=0\,;$$
$$A=C=0,\ B=74{\cdot}2 \text{ volts}\,;$$
$$B=C=0,\ A=74{\cdot}2 \text{ volts.}$$

In each case the deflection was 253·5 if the charge on the needle was positive in relation to the quadrant with which it was not connected; and was 247 when the needle was negative. This at first appeared anomalous; but the explanation is very simple. The needle is aluminium, the quadrants are either brass or brass-gilded, I am not sure which. There is therefore a contact-difference of potential between the needle and the quadrants; call it x. Thus, instead of $\theta=\frac{A^2}{2}$, we have

$$\theta=+\lambda A\left(\frac{A}{2}+x\right)$$

and

$$\theta=\lambda(-A)\left(-\frac{A}{2}+x\right);$$

this gives

$$x=\frac{6{\cdot}5}{2\lambda A}=0{\cdot}482 \text{ volt.}$$

The result was verified by using fourteen cells instead of seventy: the deflections were 10·0 and 8·8, which gives the same value to x. It is worth noting that the same cause affects the idiostatic gauge in the same way. Let the jar be charged till the gauge comes to the mark. Call P the difference of potential between the aluminium lever of the idiostatic gauge and the brass disk below which attracts it. The difference of potential between the brass of the case and the brass work of the interior is $P+x$, and between the case and the aluminium needle within the quadrants it is $P+2x$. If, however, the charge is negative, the difference is $-P+2x$. Hence the sensibility will be different from two causes, according as the jar is charged positively or negatively, till the idiostatic gauge is at its standard. For determining the constant λ we must take the mean of the two results 253·5 and 247, that is 250·25. Comparing this with the actual standard charge of the jar, and the double deflection given by one volt 172·4, when charged to the standard, we see that the irregularity has not been wholly eliminated. It appeared desirable to determine the sensibility of the instrument for a lower known charge. The charge was determined exactly as described above, and was found to be 609 volts; whilst 1 volt gave 107·1 scale-divisions double deflection; whence in the equation

$$\theta=\frac{\lambda(A-B)C}{1+kC^2},$$

we have, if

$$\lambda=0{\cdot}1816, \quad k=7\times10^{-8},$$

the following as the calculated and observed deflections:—

Calculated	250·0,	107·7,	172·4
Observed	250·2,	107·1,	172·4,

which is well within errors of observation.

This deviation from proportionality of sensibility did not appear to be worth correcting, as I was not sure that other small irregularities might not be introduced by raising the needle above the middle position within the quadrants. It appears probable that the small deviation still remaining does not arise from the attraction of the quadrants on the needle increasing the tension of the suspension, but from some cause of a quite different nature, for if it were so caused the capacity-equations would be

$$\theta_1=q_{11}A-q_{12}B-q_{33}C-q_{14}D+\alpha\theta C, \text{ \&c.,}$$

where

$$\theta=\frac{\lambda}{1+kC^2}(A-B)\left(C-\frac{A+B}{2}\right)$$

Now the experiments I have tried for determining q_{11}, q_{12}, &c., are not in accord within the limits of errors of observation, using these equations of capacity; but they are in better accord if, in lieu of the term $\alpha\theta C$, we write $\frac{\alpha\theta C}{1+kC^2}$. I have no explanation of this to offer; but in what follows it is assumed that the equations expressing the facts are

$$\left.\begin{aligned}\theta&=\mu(A-B),\text{ where }\mu=\frac{\lambda C}{1+kC^2},\\ Q_1&=\quad q_{11}A-q_{12}B-q_{14}D+\beta\mu\theta,\\ Q_2&=-q_{12}A+q_{22}B-q_{24}D-\beta\mu\theta,\\ Q_4&=-q_{14}A-q_{24}B+q_{44}D.\end{aligned}\right\}$$

We are now in a position to determine the various coefficients of capacity: in doing so it is necessary to distinguish the values of q_{11} and q_{22} when the posts by which contact with the quadrants is made are down and in contact with the quadrants, and when they are raised up out of contact; the former are denoted by $q_{11}+a$ and $q_{22}+a$, the latter by q_{11} and q_{22}, the capacity of the binding-posts being a. As a convenient temporary unit of capacity the value of $\beta\mu^2$, when the jar has the standard charge, is taken. The first set of experiments was to determine the deflections caused by known potentials with varied charge of jar, one or other of the quadrants being insulated. Three potentials of the jar were used—that of the standard indicated by the idiostatic gauge and two lower. The values of μ are denoted by μ_3, μ_2, μ_1. It was found by connecting the two quadrants to standard cells that

$$\mu_3 : \mu_2 : \mu_1 = 1 : 0{\cdot}805 : 0{\cdot}585\,;$$

and hence

$$\beta\mu_3^2=1,\ \beta\mu_2^2=0{\cdot}648,\ \beta\mu_1^2=0{\cdot}342.$$

Suppose quadrant A be insulated, and potential B be applied to quadrant B; then we have, if θ be deflection which potential B would cause with standard charge, if quadrant A were connected to the case, and ϕ the observed deflection,

$$\begin{aligned}0&=q_{11}A-q_{12}B+\beta\mu\phi\,;\\ \phi&=\mu(A-B)\,;\\ \theta&=-\mu_3B\,;\end{aligned}$$

whence

$$\phi=\theta\,.\,\frac{\mu}{\mu_3}\,.\,\frac{q_{11}-q_{12}}{q_{11}+\beta\mu^2}.$$

In the calculated values of ϕ given below,

$$q_{11}=0{\cdot}502, \qquad q_{22}=0{\cdot}543$$
$$q_{12}=0{\cdot}293, \qquad a=0{\cdot}200 \text{ for B,}$$
$$=0{\cdot}193 \text{ for A.}$$

A closer approximation to observation is obtained by assuming the two contact-posts to be of slightly different capacities; the difference given above is no more than might be expected to exist.

The jar being charged to standard potential, B was insulated, and its post raised, and A was connected to 10 Daniells, for which $\theta=1808$:—

Deflection observed $=293{\cdot}2$,
,, calculated $=293{\cdot}0$.

The post of B was lowered to contact :—

Deflection observed $=467{\cdot}0$,
,, calculated $=466{\cdot}8$.

A was now insulated and post raised, B was connected to the same battery:—

Deflection observed $=251{\cdot}0$,
,, calculated $=251{\cdot}6$.

The post of A was lowered to contact :—

Deflection observed $=429{\cdot}0$,
,, calculated $=428{\cdot}8$.

The jar was now charged to a lower potential, for which $\mu=\mu_2$ with B insulated and post raised, and A connected to 30 Daniells, for which $\theta=5468$:—

Deflection observed $=925{\cdot}0$,
,, calculated $=924{\cdot}0$.

The post of B was lowered to contact, and A connected to 10 Daniells, for which $\theta=1808$:—

Deflection observed $=470{\cdot}5$,
,, calculated $=470{\cdot}85$.

A was now insulated and post raised, B was connected to a battery of 30 Daniells, for which $\theta=5468$:—

Deflection observed = 798·0,
„ calculated = 800·0.

The post of A was lowered to contact, and B was connected to 10 Daniells ; $\theta=1808$:—

Deflection observed = 437·0,
„ calculated = 435·7.

The jar was then charged to a still lower potential, for which $\mu=\mu_1$, with B insulated and post raised, and A connected to 30 Daniells, for which $\theta=5468$:—

Deflection observed = 901·0,
„ calculated = 903·6.

The post of B was lowered to contact and A connected to 10 Daniells ; $\theta=1808$:—

Deflection observed = 437·0,
„ calculated = 438·7.

A was now insulated and post raised, and B was connected to 30 Daniells ; $\theta=5468$:—

Deflection observed = 785,
„ calculated = 792.

The post of A was lowered to contact and B connected to 10 Daniells ; $\theta=1808$:—

Deflection observed = 408,
„ calculated = 410.

The next experiment was similar, excepting only that the insulated quadrant B was connected to a condenser ; this condenser consisted merely of a brass tube insulated within a larger tube—its capacity is about 0·00009 microfarad. The jar was at its standard charge. Calling the capacity of the condenser b, in terms of our temporary unit, we have, as before,

$$\phi=\theta\frac{b+a+q_{22}-q_{12}}{b+a+q_{22}+1}\quad[\beta\mu_3^2=1].$$

When $\theta=1259$, ϕ was observed to be 927, whence $b=3·159$.

We are now in a position to obtain independent verification of the values already obtained for the constants. Suppose A be connected to the case, that condenser b is charged from a battery of known potential, such that it would give deflection θ if connected to B, and the charged condenser is then connected to B. Suppose ψ be the deflection before connection is made, ϕ after. Then

$$b(\theta-\phi)=\{q_{22}+a+1\}(\phi-\psi).$$

When $\theta=1439$ and $\psi=0$, it was found that $\phi=915$. The value of ϕ, calculated from the values of the constants already obtained, is 928.

When $\theta=1439$ and $\psi=-676$, it was found that $\phi=+676$; the calculated value is 688.

A further experiment of verification, involving only the capacity of the quadrant, is the following. The quadrant A being connected to the case, B was charged by contact instantaneously made and broken with a battery of known potential, and the resulting deflection was noted. The instantaneous contact being made by hand, no very great accuracy could be expected. Let ψ and ϕ be the readings on the scale before and after the instantaneous contact; then

$$\frac{\theta-\phi}{\phi-\psi}=\frac{1}{q_{22}+a}=1{\cdot}345.$$

The following results were obtained:—

θ.	ψ.	ϕ observed.	ϕ calculated.
1796	0	763	765
1796	-493	493	482

We next determine the coefficients q_{14} and q_{24} of induction of the induction-plate on the quadrants. This is easily done from the deflections obtained with the induction-plate, one or both pairs of quadrants being insulated. First, suppose one pair, say B, are insulated whilst A is connected to the case:—

$$0=+q_{22}\,\mathrm{B}-q_{24}\,\mathrm{D}-\beta\mu\phi,$$
$$\phi=-\mu\mathrm{B},$$
$$\theta=\mu_3\mathrm{D};$$

whence

$$\phi=\theta\frac{\mu}{\mu_3}\;\frac{q_{24}}{q_{22}+\beta\mu^2},$$

ϕ being the deflection actually observed, and θ that which the battery used would give if connected direct to the quadrants, the needle having the standard charge. When θ was 12,800 and $\mu=\mu_3$, ϕ was 418, whence $q_{24}=0{\cdot}0504$.

In the same way A being insulated but B connected to the case, ϕ was found to be 43·6, whence $q_{14}=0{\cdot}00508$.

Again, when both quadrants are insulated we have

$$\begin{aligned} 0&=q_{11}\mathrm{A}-q_{12}\mathrm{B}-q_{14}\mathrm{D}+\beta\mu\phi, \\ 0&=-q_{12}\mathrm{A}+q_{22}\mathrm{B}-q_{24}\mathrm{D}-\beta\mu\phi, \\ \phi&=\mu(\mathrm{A}-\mathrm{B}), \\ \theta&=\mu_3\mathrm{D}. \end{aligned}$$

From the first two equations,

$$(q_{11}q_{22}-q_{12}^2)(\mathrm{A}-\mathrm{B})-\{(q_{22}-q_{12})q_{14}-(q_{11}-q_{12})q_{24}\}\mathrm{D}$$
$$+(q_{22}+q_{11}-2q_{12})\beta\mu\phi=0\,;$$

whence

$$\phi=\theta\frac{\mu}{\mu_3}\,\frac{(q_{22}-q_{12})q_{14}-(q_{11}-q_{12})q_{24}}{(q_{11}q_{22}-q_{12}^2)+(q_{22}+q_{11}-2q_{12})\beta\mu^2}.$$

In the case when $\mu=\mu_3$, substituting the values already determined, we have

$$\phi=\theta\times0{\cdot}0142\,;$$

it was observed with $\theta=12{,}800$ that $\phi=183$; the calculated value would be 182.

With a lower charge on the jar, viz. when $\mu=\mu_3\times0{\cdot}805$, with B insulated, A connected to the case, and $\theta=12{,}800$, it was found that $\phi=437{\cdot}5$; the calculated value is 441.

The capacity q_{44} of the induction-plate is of no use ; its value, however, is about 0·004, in the same unit as has been so far used.

The capacity q_{33} of the needle and the coefficient of induction of the needle on either quadrant $\frac{1}{2}q_{33}$ are also of no use, but the method by which they may be obtained is worth noting. Let quadrants A be connected to the case, and let B be insulated, diminish the charge of the jar slightly by the replenisher, and suppose the consequent deflection be ϕ. Let μ and μ' be the values of μ before and after the diminution of charge, as ascertained by applying a known potential-difference between the two pairs of quadrants; we have

$$-q_{33}C = +q_{22}B - q_{33}C' - \beta\mu'\phi,$$

where

$$\phi = -\mu' B,$$

$$q_{33}(C - C') = (q_{22} + \beta\mu'^2)B,$$

which determine q_{33}, since C and C′ are known from μ and μ'.

Of course the values of the constants of an electrometer are of no value for any instrument except that for which they are determined in the state of adjustment at the time. For any particular use of the instrument it is best to determine exactly that combination of constants which will be needed. Nor is there anything new in principle in the discussion or experiments here given; they are merely for the most part the application of well-known principles to methods of using the electrometer given by Sir William Thomson himself. The method of determining the capacity of a condenser by charging it and connecting it to an insulated quadrant has been used by Boltzmann. But the invention of the quadrant-electrometer by Sir W. Thomson may be said to have marked an epoch in Electrostatics, and the instrument from time to time finds new uses. It therefore seems well worth while to make known observations made upon it in which the instrument itself has been the only object studied. Some practical conclusions may, however, be drawn from the preceding experiments. Before using the formula

$$\theta = \lambda(A - B)\left(C - \frac{A + B}{2}\right)$$

it is necessary to verify that it is sufficiently nearly true, or to determine its variation from accuracy. Unless it be sufficiently accurate through the range experimented upon, the electrometer cannot be applied by the methods well known for determining alternating potentials and the work done by alternating currents.

My pupil, Mr. Paul Dimier, has very efficiently helped me in the execution of the experiments of verification.

IV. *Notes on the Use of* Nicol's *Prism. By* James C. M'Connel, *B.A., Assistant Demonstrator at the Cavendish Laboratory, Cambridge*[*].

1. *On the Error in the Measurement of a Rotation of the Plane of Polarization caused by the Axis, about which the Nicol turns, not being parallel to the Incident Light.*

Suppose we have a beam of parallel light traversing a Nicol's prism mounted in a graduated circle. Unless we have taken special precautions, we shall find that, when the Nicol is rotated, the plane of polarization of the emergent light turns through an angle somewhat different from that measured by the circle. For instance, if the axis of rotation of the Nicol is inclined 3° to the direction of the incident or emergent light, as large an error as 1° may be made in measuring a rotation of 60°.

It is, however, tolerably well known that, as far as this error is proportional to the first power of the small angle of deviation, it may be eliminated by taking the mean of the readings in the two opposite positions of the Nicol circle separated from one another by nearly two right angles. It is of course only to a first approximation that this proportionality can be considered to hold good. And the main object of the present investigation is to determine, what is the outstanding error we are liable to in assuming this proportionality; or, in other words, with what accuracy we must adjust the Nicol circle, that this first approximation may be sufficient.

The phenomenon in its simplest form may be described thus:—If the emergent ray be parallel to the long edges of the prism it is polarized in a plane perpendicular to the principal plane of the prism, *i. e.* to the plane containing the optic axis and the normal to the face. If the ray be inclined to this position but still lie in the principal plane, then the new plane of polarization is as nearly parallel to the old one as is consistent with its containing the new ray. This indeed is obvious from symmetry. But now let the ray lean out of the principal plane: there is a marked change in the plane of polarization. It has been twisted about the ray, and the angle of twist is nearly one third of the angle of deviation. From this explanation it is easy to see in a general way how the error arises, and also how it may be eliminated by taking readings in the two opposite positions of the Nicol.

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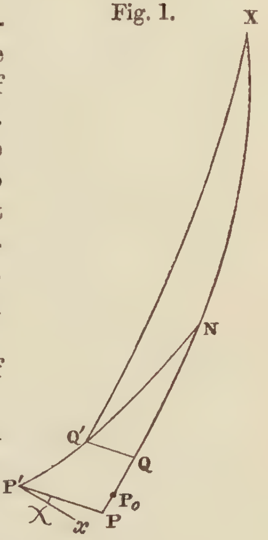
[*] Read February 28, 1885.

There is another important point that is apparent on the face of the matter. If the direction of the ray coincide with the axis of rotation, the ray will not move relatively to the prism when the prism is turned, and the rotation measured by the circle will be identical with the actual rotation of the plane of polarization. We shall find, moreover, that to the second order of approximation the final error depends entirely on the angle between these two directions (see equation 8). It is independent of any reasonably small errors in the adjustment of the Nicol in the circle.

Fig. 1.



In a Nicol's prism used as a polarizer, it is the second half of the prism that determines the plane of polarization of the emergent light. If the optic axis of the spar in the first half be not accurately parallel to the axis in the second half, the effect is not to turn the plane of polarization, but merely to mix a little ordinary light with the emergent polarized light.

The figure represents a portion of the sphere of unit radius.

X is the optic axis. N the normal to the face.

So XN is the principal plane.

P' is the wave-normal of the emergent light.

Q' is the wave-normal of the internally incident light.

Draw $Q'Q$, $P'P$ perpendicular to XN produced. Join XQ'.

Let $P'x$ be the plane of polarization of P' and $PP'x=\chi$. Then χ is a function of $P'P$ and PN.

The exact function would be a very complicated expression, so we shall content ourselves with an approximation. $P'P$ is always small, and P lies near some fixed point, P_0 say, in the plane XN. Let

$$P'P=\alpha, \quad PP_0=\beta.$$

We shall reject the cubes and higher powers of α and β and limit ourselves to expanding χ as far as the squares. We know that when α is zero, χ is zero; so our expression can only contain the terms α, α^2, and $\alpha\beta$.

In a Nicol's prism it is the extraordinary ray which emerges, so the plane of polarization of Q′ is perpendicular to Q′X. Let us assume for the present that the plane of polarization of P′ makes the same angle with the plane of incidence P′Q′N as does the plane of polarization of Q′. The plane of polarization of Q′ makes with P′Q′N an angle 90°−NQ′X. Hence

$$\chi=90^\circ-NQ'X-NP'P. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But in the triangle NP′P, P is a right angle; so we have

$$\tan(90^\circ-NP'P)=\cot NP'P=\frac{\sin P'P}{\tan NP}=\frac{\alpha}{\tan NP}, \quad . \quad (2)$$

neglecting cubes.

It remains to find NQ′X.

$$NQ'X=XQ'Q-NQ'Q,$$

$$\therefore \quad \chi=90^\circ-NP'P+NQ'Q-XQ'Q. \quad . \quad . \quad . \quad (3)$$

By spherical triangles,

$$\left.\begin{aligned}\tan(90^\circ-XQ'Q)&=\frac{\sin Q'Q}{\tan XQ},\\ \tan(90^\circ-NQ'Q)&=\frac{\sin Q'Q}{\tan NQ}.\end{aligned}\right\}$$

Also

$$\left.\begin{aligned}\sin Q'Q&=\sin Q'N\sin N,\\ \sin P'P&=\sin P'N\sin N.\end{aligned}\right\}$$

But $\sin P'N=\mu\sin Q'N$, where μ is the extraordinary index of refraction for that wave, which we may take to be constant.

$$\therefore \sin Q'Q=\frac{1}{\mu}\sin P'P=\frac{\alpha}{\mu} \text{ to our order of approximation.}$$

Hence

$$\left.\begin{aligned}\tan(90^\circ-XQ'Q)&=\frac{\alpha}{\mu}\frac{1}{\tan XQ},\\ \tan(90^\circ-NQ'Q)&=\frac{\alpha}{\mu}\frac{1}{\tan NQ}.\end{aligned}\right\} \quad . \quad . \quad . \quad (4)$$

Since tan XQ and tan NQ occur in small terms, we may reject squares in finding their values.

In NQ let us take a point Q_0 such that $\sin NP_0 = \mu \sin NQ_0$. Then, since $\sin(NP_0 + \beta) = \mu \sin(NQ_0 + QQ_0)$,

$$\beta \cos NP_0 = \mu QQ_0 \cos NQ_0,$$

and

$$\left.\begin{aligned} \frac{1}{\tan XQ} &= \frac{1}{\tan XQ_0} - \frac{QQ_0}{\sin^2 XQ_0} = \frac{1}{\tan XQ_0} - \frac{\beta \cos NP_0}{\mu \cos NQ_0 \sin^2 XQ_0} \\ \frac{1}{\tan NQ} &= \frac{1}{\tan NQ_0} - \frac{QQ_0}{\sin^2 NQ_0} = \frac{1}{\tan NQ_0} - \frac{\beta \cos NP_0}{\mu \cos NQ_0 \sin^2 NQ_0}. \end{aligned}\right\} \quad (5)$$

Now in the expansion of the tangent of a small angle the squares of the angle do not appear. So we have by (3), (2), and (4),

$$\chi = \frac{\alpha}{\tan NP} + \frac{\alpha}{\mu}\frac{1}{\tan XQ} - \frac{\alpha}{\mu}\frac{1}{\tan NQ},$$

while

$$\frac{1}{\tan NP} = \frac{1}{\tan NP_0} - \frac{\beta}{\sin^2 NP_0}.$$

So by (5),

$$\begin{aligned} \chi = \alpha &\left\{ \frac{1}{\tan NP_0} + \frac{1}{\mu \tan XQ_0} - \frac{1}{\mu \tan NQ_0} \right\} \\ + \alpha\beta &\left\{ -\frac{1}{\sin^2 NP_0} - \frac{\cos NP_0}{\mu^2 \cos NQ_0}\left(\frac{1}{\sin^2 XQ_0} - \frac{1}{\sin^2 NQ_0} \right) \right\} \quad (6) \\ = l\alpha &- m\alpha\beta \text{ say.} \end{aligned}$$

In Nicol's prism, as it is usually cut,

$$\begin{aligned} NX &= 41\tfrac{1}{2}^\circ, \\ NP_0 &= 22^\circ, \\ \mu &= 1{\cdot}54, \\ \therefore \quad NQ_0 &= 14^\circ. \end{aligned}$$

Substituting these values, we obtain

$$l = {\cdot}32, \quad m = {\cdot}84;$$

$$\chi = {\cdot}32\,\alpha - {\cdot}84\,\alpha\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

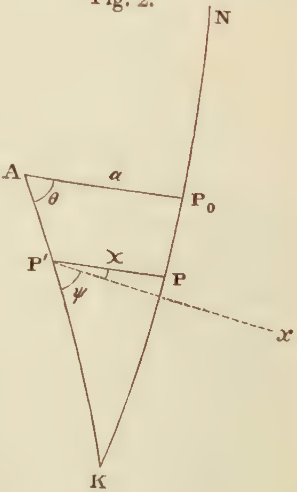
In the foregoing we have assumed that there is no rotation of the plane of polarization on refraction out of the spar, or, in other words, that the angle between the plane of polarization and the plane of incidence remains unchanged. There is of course a change, but it is merely due to the disturbing effect of reflection and is very small. If we treat the spar as a homogeneous medium and use Fresnel's formulæ, we find the rotation is about $\cdot 005\,\alpha$ in the case of an ordinary Nicol.

I have calculated from Neumann's theoretical formulæ for refraction at the surface of crystals what the rotation would be in the case of light entering the Nicol and exciting only the extraordinary wave, and find it is less than with a homogeneous medium. In the case of light leaving the Nicol there are two reflected waves, so the formulæ would probably be very complicated. I think we may safely assume that the rotation is less than $\cdot 01\alpha$, and may therefore be neglected. In the case of a flat-ended Nicol the incidence is nearly direct, and the rotation may obviously be neglected.

Owing to the circumstance that this rotation is negligible, the whole of this investigation applies without alteration to the case when the Nicol is used as an analyzer, and the light is consequently incident on the spar. In this case we have to find the position of the plane of polarization that only the ordinary wave may be excited, and the formula for the rotation is the same as in an isotropic medium.

We are now in possession of a convenient formula for expressing the position of the plane of polarization in terms of the direction of the emergent light relative to fixed planes in the Nicol. We shall apply this formula to the discussion of the error in measuring a rotation. This part of the investigation would be very short if we confined ourselves to the first approximation. A number of complications are introduced by the necessity of retaining the squares and products of small quantities.

Fig. 2.



The figure represents a portion of the sphere of unit radius. Let

A be the axis of rotation,

N P the principal plane,

P′ the emergent wave-normal,

P′x the plane of polarization.

Draw A P_0 and P′ P perpendicular to the principal plane. This fixes P_0, which has hitherto been arbitrary within certain limits. A and P′ are fixed in space, while N and P_0 are rotated round A.

For simplicity, suppose the circle-reading is zero when P′ lies on A P_0. Then $\theta = P_0AP'$ is the reading at any time.

Produce A P′ to meet N P_0 in K.

What we want to measure is the rotation of $P'x$ round P′; so let $xP'K = \psi$. As before, let $P'P = \alpha$, $PP_0 = \beta$, and let $AP_0 = a$, $AP' = r$. Then α, β, a, r are all small. Since the figure A P_0 P P′ is small, we have to our order of approximation

$$\left.\begin{aligned} \alpha &= a - r\cos(\psi+\chi), \\ \beta &= r\sin(\psi+\chi). \end{aligned}\right\}$$

Let us now find the difference between θ and $\psi+\chi = PP'K$.

By spherical triangles,

$$\left.\begin{aligned} \cos K &= \sin(\psi+\chi)\cos PP', \\ \cos K &= \sin\theta\cos a; \end{aligned}\right\}$$

$$\therefore\ \sin(\psi+\chi)\cos PP' = \sin\theta\cos a,$$

$$\sin\theta = \sin(\psi+\chi)\left(1 - \frac{\alpha^2}{2} + \frac{a^2}{2}\right),$$

$$\sin(\psi+\chi) + (\theta-\psi-\chi)\cos(\psi+\chi) = \sin(\psi+\chi)\left(1 - \frac{\alpha^2}{2} + \frac{a^2}{2}\right);$$

$$\therefore\ \theta-\psi-\chi = \frac{a^2-\alpha^2}{2}\tan(\psi+\chi)$$

$$= \frac{a^2-\alpha^2}{2}\tan\psi, \text{ neglecting cubes.}$$

But

$$a^2-\alpha^2 = 2ar\cos\psi - r^2\cos^2\psi,$$

and

$$\begin{aligned} \chi &= l\alpha - m\alpha\beta \\ &= l(a - r\cos\overline{\psi+\chi}) - mr\sin\psi(a - r\cos\psi) \\ &= l\{a - r\cos\psi + lr\sin\psi(a - r\cos\psi)\} - mr\sin\psi(a - r\cos\psi) \\ &= la - lr\cos\psi + (l^2-m)r\sin\psi(a - r\cos\psi); \end{aligned}$$

$$\begin{aligned} \therefore\ \theta = \psi + la - lr\cos\psi &+ (l^2-m+1)ar\sin\psi \\ &- (l^2-m+\tfrac{1}{2})r^2\sin\psi\cos\psi. \end{aligned}$$

Now let the plane of polarization be turned through 180° and the new reading of the Nicol circle be $180° + \theta_1$. Instead of ψ we must write in the last formula $180+\psi$. So

$$\begin{aligned} \theta_1 = \psi + la + lr\cos\psi &- (l^2-m+1)ar\sin\psi \\ &- (l^2-m+\tfrac{1}{2})r^2\sin\psi\cos\psi; \end{aligned}$$

$$\therefore \frac{\theta+\theta_1}{2} = \psi + la - (l^2 - m + \tfrac{1}{2}) r^2 \sin\psi \cos\psi,$$

$$\frac{\theta_1-\theta}{2} = lr \cos\psi - (l^2 - m + 1) ar \sin\psi.$$

Inserting numerical values, we have

$$\frac{\theta+\theta_1}{2} - \psi = \cdot 32a + \cdot 24 r^2 \sin\psi \cos\psi, \quad . \quad . \quad . \quad (8)$$

$$\theta_1 - \theta = \cdot 64 r \cos\psi, \text{ approximately.} \quad . \quad . \quad . \quad . \quad (9)$$

To render these formulæ intelligible to one who has not read through the investigation, we may remark that we have to deal with one plane fixed in space, viz. the plane containing the axis of rotation of the Nicol circle and the direction of the emergent light.

θ and $180^\circ + \theta_1$ are the readings of the circle in the two opposite positions, and $\theta = 0$ when the principal plane of the Nicol is at right angles to the fixed plane.

ψ is the angle between the plane of polarization and the fixed plane.

a is the angle between the axis of rotation and the principal plane.

r is the angle between the axis of rotation and the emergent light.

The angles are supposed to be expressed in circular measure in these as in all the other formulæ.

The first term on the right-hand side of equation (8) is a constant, and is therefore of no consequence in measuring a change of ψ. The second term is a measure of the outstanding error.

Let us suppose

$$r = 3^\circ, \quad \psi = 30^\circ.$$

Then

$$\cdot 24 r^2 \sin\psi \cos\psi = 1';$$

but if $\psi = -30^\circ$,

$$\text{the same} = -1'.$$

So in measuring a rotation of 60° we may be subject to an error of 2′ if the axis of rotation be inclined to the emergent light at an angle of 3°. To make sure that the error shall be less than 1′, the last-mentioned angle must be kept within 2°.

Equation (9) affords the means of deducing the value of r from the difference of readings in the two opposite positions.

The second term of (8) is due to two main causes. One is that the rotation of the plane of polarization takes place about one axis, while the rotation is measured about another. The effect of this appears in the coefficient $-\frac{1}{2}$. The other part of the error is due to the values of χ in the two opposite positions not exactly neutralizing each other. This appears in the coefficient $m-l^2$. If these two causes had reinforced instead of counteracting one another, the resultant error would have been five times as large.

If the arrangements permit of taking readings when the plane of polarization is inclined at 90° to its two first positions, we can, by taking the mean of all four readings, get rid of the error depending on the squares of small quantities, as is evident from (8). This is what Lord Rayleigh has done in his recent measurements of the electromagnetic rotation in bisulphide of carbon.

The formula (6) applies to almost every mode of cutting a Nicol, except the important case when the ends are cut off square to the length, the case of the so-called "flat-ended Nicol." This case requires the investigation to be modified. The peculiarity consists in N lying very close to P′.

$$\chi = 90° - NP'P + NQ'Q - XQ'Q,$$

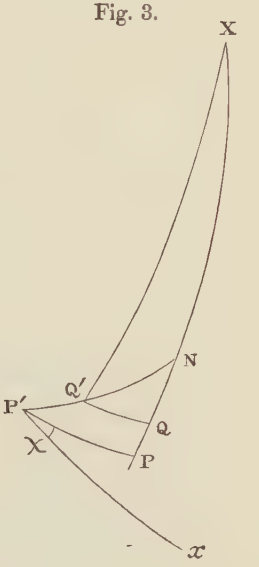
as before;

and by (4) and (5),

$$\tan(90° - XQ'Q) = \frac{\alpha}{\mu}\frac{1}{\tan XQ_0} - \frac{\alpha\beta}{\mu^2 \sin^2 XQ_0}.$$

If we take N as the point from which to measure β, P_0 and Q_0 will coincide with N.

Fig. 3.



We have now to find $NP'P - NQ'Q$. The process is precisely similar to one already performed in finding equation (8);

and we obtain

$$\mathrm{NP'P}-\mathrm{NQ'Q}=\frac{\mathrm{PP'}^2-\mathrm{QQ'}^2}{2}\tan\mathrm{NP'P}$$

$$=\frac{\mu^2-1}{2\mu^2}\alpha\beta;$$

$$\therefore\ \chi=\frac{\alpha}{\mu\tan\mathrm{XN}}-\frac{\alpha\beta}{\mu^2\sin^2\mathrm{XN}}-\frac{\mu^2-1}{2\mu^2}\alpha\beta$$

$$=l\alpha-m\alpha\beta,\ \text{say}.$$

Taking $\mathrm{XN}=63°$, $\mu=1{\cdot}52$, we obtain

$$l=\cdot33,\quad m=\cdot83.$$

So the numbers are almost exactly the same as in the case of the ordinary Nicol.

As a test of the accuracy of the above calculations, I made a few observations on a flat-ended Nicol. I found that, turning the Nicol through $30'$, about an axis in the principal plane and at right angles to the incident light, turned the plane of polarization through $11'\pm1'$.

In a paper on Polarizing Prisms (Phil. Mag. 1883) Mr. Glazebrook has suggested a new form of flat-ended prism in which the axis of the spar is at right angles to the length of the prism. In this case $\mathrm{XN}=90°$, and $\mu=1{\cdot}49$.

$$\therefore\ l=0,\quad m=\cdot73,$$

$$m-l^2-\tfrac{1}{2}=\cdot23.$$

So instead of equation (8) we have

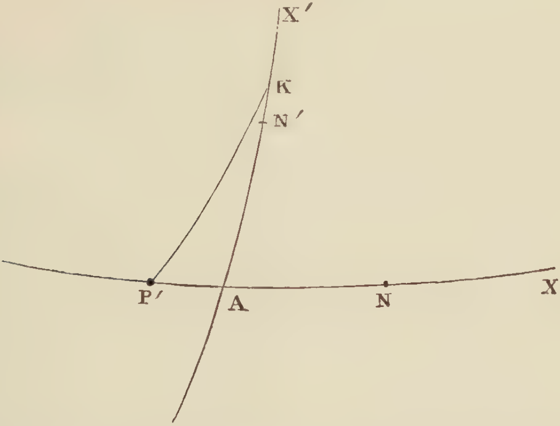
$$\frac{\theta+\theta_1}{2}-\psi=\cdot23r^2\sin\psi\cos\psi,$$

and the outstanding error is substantially the same as before.

Equation (9) appears to be inconsistent with a result obtained by Mr. Glazebrook (Phil. Mag. Oct. 1880, p. 252). For the sake of simplicity, he supposed the ordinary ray only to emerge from the crystal. He examined two positions of the Nicol. First the emergent ray lay in the principal plane, so that the plane of polarization coincided with the principal plane. Next he supposed the Nicol to be turned through 90° about an axis, lying in the principal plane, but inclined at 5° to the emergent ray, and he found that now the plane of polarization was inclined at 5° 3′ to the principal plane. *But this does not show that the plane of polarization has been turned through*

$90° \pm 5°\ 3'$ *about the direction of the ray*, as is evident on examination of the annexed figure.

Fig. 4.



P′ is, as before, the emergent light; N, N′, X, X′ are the two positions of the normal to the face and the axis.

In the first position the plane of polarization is P′ A N X.

In the second position let it be P′ K.

Mr. Glazebrook finds P′KA = 5° 3′.

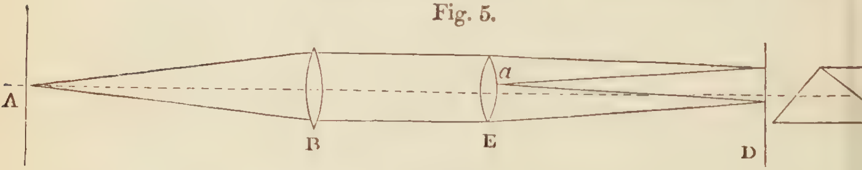
The angle, however, that we wish to find, is the angle through which the plane of polarization has been turned about the direction of the ray, viz. A P′ K, and this, of course, is not equal to 90° − P′ K A.

After I had written the greater part of this paper, I found that the ground had been already traversed by Sande Bakhuyzen (Pogg. *Ann.* cxlv. p. 259, 1872). His investigation is considerably longer than mine, and he only examines the case of the ordinary Nicol. He obtains results of the same general form; but he falls into at least one serious error; so his numerical values are quite different. Instead of assuming, as I have done, that the rotation of the plane of polarization on refraction out of the crystal is negligible, he uses a complicated formula, giving directly the position of the plane of polarization of the incident ray in which only the extraordinary ray is excited in the crystal. This formula he has apparently deduced from a formula given by Neumann (Pogg. *Ann.* xlii. p. 11). And it is here that the error occurs. He has not paid sufficient attention to Neumann’s explanation of how the quantities in the formula are to be measured. He puts β for ω; whereas he should put $\beta-\pi$ for ω. Making this correction in his equation (5), we find all the terms in the denominator are of one sign, and the value of tan A is considerably diminished. We find, too, that $90°-A$ is practically the same as the angle between the plane of incidence and the plane of polarization in the crystal, which is what I have assumed.

In this very same operation Bakhuyzen falls into another mistake. He is considering the case of light emerging from the Nicol, while he uses the formula for light entering the crystal. Even with glass this would have been wrong; but with crystal the cases are totally distinct, for when light is refracted out of the crystal there are two reflected rays instead of only one.

It may not be out of place here to describe a simple method for testing the necessary adjustment, which I lately found convenient. In my arrangement (fig. 5) the source of light was an illuminated slit A, whence the light passed through a lens B, placed so that the slit was at its principal focus. Thus a parallel beam of light fell on the polarizing Nicol C. To the Nicol circle I attached a plane mirror D as nearly as possible at right angles to the axis of rotation. Between B and D, I interposed a lens E at a distance of half its focal length from D. An image of the slit is thus formed at a on the surface of the lens E, or rather on a scrap of paper attached to it. The image a of course remains at the same point on the paper, however the lens E is moved laterally in its own plane. On rotating the Nicol circle I found that a did not move; so I knew that the mirror was at right angles to the axis of rotation. I then adjusted the Nicol circle till the middle point of the image of the slit fell on the optical centre of the lens, which I had previously marked on the paper. If I had

Fig. 5.



chosen a wrong point as the optical centre, the error would have been at once evident on turning the lens E through 180° in its own plane. This method gave without much difficulty an accuracy of 1°, with a lens E of only 4 inches focal length.

2. *On a new Method of obtaining the Zero-reading of a Nicol Circle.*

In a large class of experiments on polarized light, it is necessary to know the reading of the Nicol circle when the plane of polarization is parallel to the axis of rotation of some part of the apparatus, *e. g.* of the table of a spectrometer. The usual method of obtaining this reading depends on the polarizing power of a glass reflecting surface. It is easy to fix the surface on the spectrometer-table parallel to the axis of rotation. If the angle of incidence be made that of maximum polarization, and the Nicol turned till the reflected light is reduced to a minimum, the plane of polarization is then perpendicular to the plane of incidence. This method is simple, but it is not very sensitive. Even with poor illumination the minimum reflected light is, with glass at any rate, by no means evanescent; and as the Nicol is turned the intensity remains sensibly constant for some distance on either side of the minimum point. The sensitiveness, too, does not increase, but rather falls off with a more powerful light.

The method I am about to describe is nearly as simple; and, as it depends on the crossing of two Nicols, its sensitiveness is only limited by the power of the source of light. In its simplest form the process is as follows:—An auxiliary Nicol is fixed on the table of the spectrometer, the polarizer turned till the light is quenched, and the reading taken. Then the table of the spectrometer is turned through two right angles—the axis of rotation having been previously set perpendicular to the incident light—the light quenched, and the reading taken again. *The plane of polarization now leans as much to one side of the axis as it did before to the other.* So the mean of the two readings is the reading when the plane of polarization is parallel to the axis of rotation of the table.

The above is of course only a general explanation. The statement in italics requires fuller examination, and we shall find that it is only strictly true when the Nicol is symmetrical. Let us, then, first suppose that the Nicol is perfectly symmetrical—that is, that the two faces are parallel and the optic axes of the two halves coincident. We shall assume throughout the investigation that the axis of rotation is accurately perpendicular to the incident light. In the previous note we have carefully determined the position of the plane of polarization in terms of the direction of the emergent light, and we have shown that, to a very close approximation, it is at right angles to that position of the plane of polarization of light incident along the same path which gives complete extinction in the Nicol. The discrepancy was shown to be negligible. So there is no ambiguity in speaking of the plane of polarization of either half of the Nicol for a particular direction of the light.

In the first position of our Nicol on the spectrometer, the emergent light is parallel to the incident, and the planes of polarization of the two halves are also parallel. The axis of rotation, too, is at right angles to the incident light. Imagine, then, a line fixed relatively to the Nicol to be drawn from the second face parallel to the emergent light. When the Nicol has been rotated through two right angles, this line becomes again parallel to the incident light, and the plane of polarization of the second half of the Nicol now leans exactly as much to one side of the axis of rotation as the plane of polarization of the first half leant before to the other. The process therefore is so far strictly accurate.

Secondly, let us suppose that the Nicol, though not symmetrical, yet produces no angular deviation on light traversing it. We have seen in the previous note that the plane of polarization of a Nicol is nearly at right angles to the principal plane, and that to a first approximation the small deviation is proportional to the angle between the external light and the principal plane.

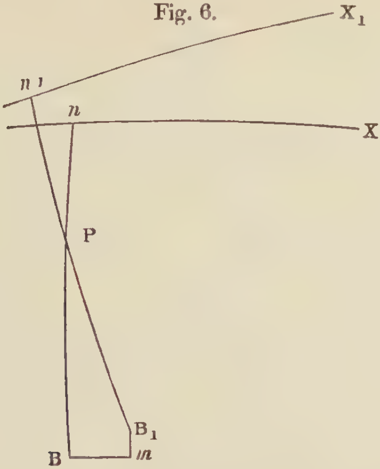
Let X n, $X_1 n_1$ be the principal planes of the two halves.

B, B_1 their poles.

P the direction of incident and emergent light.

Draw B m at right angles to B P and $B_1 m$ at right angles to B m. We may assume that B B_1 is very small and B P nearly a right angle. The angle between the planes of polarization of the two halves is composed of two parts, of which one, B P B_1, is approximately equal to B m, and the other is approximately proportional to the difference between P n_1 and P n and therefore to B_1m. So the angle is independent of small variations in the position of P, and is therefore a constant for each particular Nicol. In some Nicols, as we go from one half to the other, the plane of polarization is turned in the direction of a right-handed screw. These we may call right-handed Nicols with a certain rotation-angle. Supposing, then, our auxiliary Nicol is right-handed, the plane of the second half leans at first too much to the right, but, when the Nicol has been rotated through 180°, it leans too much to the left. The correct reading in the case of a right-handed Nicol is therefore that obtained by turning the Nicol circle from the mean reading through half the rotation-angle in the right-handed direction.

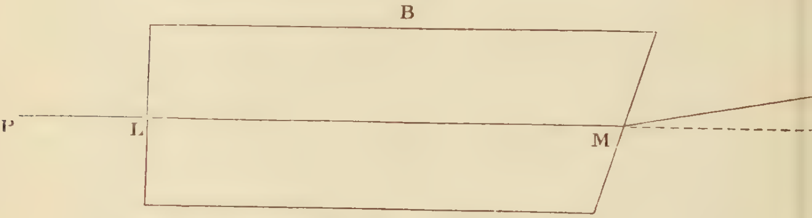
Fig. 6.



The question now arises, how we are to determine the rotation-angle. The following process explains itself. Place the auxiliary Nicol B on the path of the light from the polarizer A, which is mounted in a graduated circle. Turn A till the light is quenched and take a reading. Then turn A so as to allow the light to pass through B, and place in the path of the light from B a third Nicol C. Adjust C till the light from B is quenched. Remove B, cross C with A, and read again.

So far the matter is tolerably simple, but if the Nicol produce deviation of the light complications are introduced.

Fig. 7.



The figure represents a flat-ended Nicol B traversed by a ray of light P L M P_1. The argument, however, is perfectly general. A small error is due to the Nicol C having to deal first with light parallel to M P_1 and afterwards with light parallel to P P′. This may be eliminated by taking two readings, the Nicol B in the interval having been turned through two right angles or thereabouts about L M. The mean thus found gives us the position of the plane of polarization of P L, which is most nearly parallel to the plane of polarization of M P_1. We know nothing about the plane of polarization of light which issues parallel to M P′. Clearly therefore, when our Nicol is mounted on the spectrometer-table, we ought not to turn it exactly through 180°, but we ought to turn it till the incident light is related to the face M in the same way as P_1 M was at first.

We have then this practical rule. Place the auxiliary Nicol on the spectrometer in such a position that the deviation lies wholly in the plane of rotation. To get the second reading, turn the table in the direction of the deviation through an angle equal to 180° less the deviation. Correct the mean of the two readings by half the rotation-angle as before.

Thus we are able to get an accurate value of the zero-reading in spite of the dissymmetry of the Nicol and the deviation of the light, provided only the two faces of the Nicol are good planes. We have been compelled, however, to limit ourselves to first approximations. On this point it is to be noticed that the outstanding error is due to two causes, the inaccuracy of adjustment of the auxiliary Nicol, and the dissymmetry of the Nicol itself. The latter we may assume to be very small, while the former has very slight effect. It only comes in as a secondary cause, for if the dissymmetry were removed the outstanding error would be zero. Hence it is not necessary in general to take elaborate precautions about the setting of the Nicol in the determination either of the zero-reading or of the rotation-angle. If great accuracy be required, it will be sufficient to attach permanently to the Nicol a small reflecting surface, and make this always perpendicular to the incident light. The planes of polarization of both halves will then be perfectly definite.

I have treated the method as applied to a Nicol used as polarizer, but it is by no means restricted to this case. The polarizer may be any kind of polarimeter, or if necessary the polarimeter may be the analyzer, while the auxiliary Nicol is used as polarizer. The method occurred to me when I was engaged on some observations on the refraction of polarized light at the surface of Iceland spar. I will describe the arrangements in so far as they bear on the matter in hand; for, as it happened, they gave a very high degree of accuracy in the determination of the zero-reading.

As my source of light I employed part of the filament of a Swan incandescent lamp. I selected a straight piece, and cut off the light from the rest of the filament with a diaphragm placed immediately in front of the lamp. The filament was placed at the principal focus of a lens, from which the light passed to the polarizing Nicol. Then came the spectrometer, from which the collimator had been dismounted; so only the telescope remained. When the telescope was focused for infinity and directed towards the light, a sharply defined image of the selected portion of the filament was seen. The top and bottom limits of the image were not hard lines, owing to the diaphragm not being quite at the principal focus of the collimating lens. They were, however, sufficiently definite for practical purposes.

It was necessary for my other observations that the axis of rotation of the spectrometer should be set at right angles to the light coming from the middle of the filament. I accomplished this in the following manner. I first mounted a reflecting surface on the table of the spectrometer, and adjusted it to be parallel to the axis of rotation. This may be done with accuracy although the telescope be tilted. Next I adjusted the axis of the telescope to be at right angles to this surface, by getting the cross wires to coincide with their image formed by reflection at the surface. Then I removed the reflecting surface, directed the telescope towards the light, and tilted the spectrometer till the cross wires coincided with the middle of the image.

When the auxiliary Nicol was mounted on the spectrometer-table and the polarizer turned past the crossed position, the image of the filament by no means entirely disappeared ; but a patch of nearly complete extinction moved down the image from the top to the bottom. This was of course to be expected, since different positions of the image correspond to different directions through the Nicols, and therefore to different positions of the two planes of polarization. The reading was taken when the patch was halfway down the slit. This method of reading proved to be very sensitive. Without any special care I could get a number of successive readings whose greatest difference was 1′. This was quite sufficient for my purposes. Indeed my verniers were only graduated to 1′. But I am convinced that a far higher degree of sensitiveness could be reached if desired, by proceeding on the same lines.

In securing great accuracy two precautions become very important. The source of light should be of uniform brightness. Here a well-made carbon filament is nearly perfect. Again, as much light should reach the eye from the top of the filament as from the bottom, abstraction being made of course of what is stopped by the polarizing properties of the Nicols. For this it is convenient that the lateral throw of the Nicols should be as small as possible. Flat-ended Nicols are therefore to be preferred. But if a pair of Nicols were made for the purpose, it would be easy so to slant the faces that the extraordinary ray should go straight through, and there should be no lateral throw.

The incandescent lamp is a more powerful source of light for this purpose than is perhaps at first sight apparent. For it may be shown that with a given slit and object-glass the intensity of the illumination of the surface of the first Nicol depends solely on the intrinsic brightness of the source of light. The filament is practically a slit with the source of light brought into contact with it, so there is no difficulty about the whole of the surface of the Nicol being illuminated, and the illumination is just as powerful as would be given by a glowing sheet of the same brightness half an inch broad placed a little distance behind a slit. The intrinsic brightness of the filament of a Swan lamp is many times greater than that of a good gas-flame viewed edgeways. Besides, even if the gas-flame be placed very close to the slit, it is only the nearer portion of the flame that illuminates the whole surface of the Nicol. The further portion only lights up a narrow strip in the middle of the surface.

I used two flat-ended Nicols, belonging to Mr. Glazebrook, of about half an inch aperture. One of these, which gave an angular throw of 15′, had a left-handed rotation-angle of 38′.

I find that the plan of reading by means of the motion of the patch of extinction has been the subject of an elaborate paper by Lippich (*Sitzb. der kais. Akad. der Wissensch. Wien,* Febr. 1882).

V. *On some new Forms of Polarizing-Prism.*
By Mr. H. G. MADAN[*].

MR. MADAN exhibited and described some new forms of polarizing-prisms. The first of these is by M. Bertrand, and has been described by him (*Comptes Rendus,* September 29, 1884). The prism consists of a parallelopiped of dense flint glass of refractive index 1·658, the same as that of Iceland spar for the ordinary ray. The glass prism is cut like the spar of a Nicol's prism, a cleavage-plate of spar being cemented between the two halves by an organic cement of refractive power slightly greater than 1·658. A beam of light traversing the prism is incident upon the spar at an angle of 76° 44′. The ordinary ray passes through without change; but the extraordinary ray is totally reflected at the

[*] Read February 28, 1885.

first surface. The prism gives a field of 40°. M. Bertrand's prism has the great advantage of requiring only a very small quantity of Iceland spar, a substance that is becoming very scarce and expensive. The other prisms shown were: a similar one by M. Bertrand, described in the same paper; a double-image prism by Ahrens, described in the Phil. Mag. for January 1885; and a modification of the latter by Mr. Madan, described in 'Nature' for February 19.

VI. *The most Economical Potential-difference to employ with Incandescent Lamps. By* Professors W. E. AYRTON, *F.R.S., and* JOHN PERRY, *M.E.*[*]

THE subject in connection with which the accompanying paper is a small contribution is one of considerable commercial importance. It has long been known that the luminous power of an electric lamp increased much more rapidly than the power electrically expended on it; or, that the number of candles per horse-power increased as the lamp was made to become brighter and brighter. Perhaps the earliest experiments on the subject were those published by Sir William Thomson in 1881, and those made by our students in 1880. Since that time (that is, during the last five years) tests of the efficiency of various types of incandescent lamps have been made over and over again by various persons, without perhaps its being clearly realized that such efficiency-experiments by themselves gave us no idea of the commercial value of any particular lamp.

It is not sufficient to know that when a lamp is giving out a certain number of candles it absorbs so much power per candle, and when giving out a much larger number of candles it absorbs so much less power per candle; but what must be known in addition is the life of the lamp at each of these two candle-powers, before we can decide whether it is more economical to use the lamp with the filament at not much more than a dull red or when brilliantly luminous with a bluish tint.

[*] Read February 28, 1885.

For with the filament at a comparatively low temperature, although the efficiency of the lamp is low, its life will be great; whereas if the temperature of the filament be high, the large efficiency will be to some extent balanced by its short life and the consequent large cost due to lamp renewals.

In the 'Electrician' for January 31st of this year M. Foussat gives a table of values of the life of a 100-volt Edison lamp for different potential-differences; and M. Foussat has assured us that these numbers were obtained experimentally, and not from calculation according to any theory. That the results should lie so nearly in a curve is undoubtedly at first sight a little striking; but if it be remembered that they are stated to represent the results obtained from the *averages* of a very extended series of tests, their great regularity is not more striking than is a curve showing graphically the result of mortality tables. We asked M. Foussat whether he had also the results of experiments on the efficiency of the same type of 100-volt lamps, as our own experiments on the efficiency of Edison lamps had been made with 55-, 108-, and 110-volt lamps. As, however, he had no such results, and as his tests of lives are the only ones that we now have, and possibly may have, until the completion of the excellent work at present being carried out by the Electric-Lighting Jury of the Health Exhibition, we commenced this investigation by endeavouring to combine the lives given by M. Foussat with the results previously published for efficiency. To enable us to do this, we assumed that the life of 1000 hours given by M. Foussat for his 100-volt lamps when used with a potential-difference of 100 volts would be the same as that for our 108-volt lamp when employed with a potential-difference of 108 volts; or, in other words, we regarded lives given by him as applicable to our 108-volt Edison lamps when each of his potential-differences was multiplied by $\frac{108}{100}$.

To ascertain the most economic potential-differences, we must proceed as follows:—Let $f(v)$ be the life in hours as a function of the number of volts constantly kept on the lamps, $\theta(v)$ the number of candles emitted by one lamp as a function of the potential-difference in volts employed, let p be the price in pounds paid for one lamp, and n the number of hours per year that a lamp is kept burning ; then

$$\frac{p \times n}{f(v) \times \theta(v)}$$

stands for the cost per year per candle, as far as the renewal of lamps is concerned.

Next, let H stand for the cost in pounds of an electric horse-power per year for the number of hours an electric horse-power is employed. It is often assumed that H is proportional to the number of hours per day that the power is used, or that the yearly bill for power should be based on the horse-power hours, or total energy consumed ; but this idea, which runs even through the Electric Lighting Act, is quite an erroneous one. H will be of the form $h + \text{F}(n)$, where h is a constant independent of the number of hours and depending on the rent of the site, capital expended upon engines, dynamos, leads, &c.; and $\text{F}(n)$ is some function of the number of hours during which electric power is required, and depends on the cost of coal, superintendence, &c. If the light were only required for one or two hours in a district where rent was very high, h would be the all-important term and $\text{F}(n)$ would be unimportant; whereas if the electric power were required for various purposes, for, say, 15 or 20 hours out of the 24, in a place where rent was low but coal dear, $\text{F}(n)$ would be the important item in the yearly bill for electric power supplied.

Let $\phi(v)$ be the watts per candle, expressed as a function of the number of volts employed at the terminals of a lamp; then

$$\frac{\text{H}}{746} \times \phi(v)$$

represents the cost per year per candle as far as the production of electric power is concerned.

The total cost, therefore, per year per candle is

$$\frac{p \times n}{f(v) \times \theta(v)} + \frac{\text{H}}{746}\phi(v) \text{ pounds,} \quad . \quad . \quad . \quad (\text{A})$$

and we must find the value of v that makes this a minimum.

There are two ways in which such a problem can be solved: the one a graphical method, the other an analytical method.

The former may be used by even elementary students, and will be given first. It consists in drawing curves to represent, 1st, $f(v)$ in terms of v, 2nd, $\theta(v)$ in terms of v, and 3rd, $\phi(v)$ in terms of v; and from these the values of $f(v)$, $\theta(v)$, and $\phi(v)$ are each determined graphically for many values of v, and the value of A calculated for each of these values. A fourth curve is then drawn, connecting the values of A with those of v, when it is easy to see by inspection for what value of v the expression A has a minimum value.

The following is the result so obtained for the 108-volt Edison lamps used for lighting the Finsbury Technical College :—

p is taken at 5*s.*, or £0·25 ;

n as 560 hours, the time per year, approximately, during which the lamps are lighted ;

H as £5 : this is perhaps a rather high estimate for the cost of power, considering that the interest on the steam-engine, dynamos, &c., price of coal burnt, wages of the engine-driver and stoker have to be mainly debited to the driving of the College workshops, supplying power for the dynamos worked for experimental purposes ; but it will be accurate enough to take the sum of £5 per year as representing yearly interest on extra plant and the yearly interest on the extra cost of supplying one electric horse-power during the 560 hours.

Then the value of v which makes A a minimum turns out to be about **106** *volts*; and on account of the flatness of the curve connecting the expression A with v, we see that in this particular case the annual cost of supplying light is only increased by 3·5 per cent., if the potential-difference between the mains be kept diminished to about 104·8 or kept up to about 108 volts. Also, that keeping the potential-difference diminished to about 104·5, or increased to about 108·5 volts, increases our total annual cost of lighting by 5 per cent.

More recently one of our students, Mr. Robertson, has been making efficiency-experiments on some Edison lamps obtained from France ; and by trial he has found some that give 16 candles at about 100 volts, and are therefore presumably of the same type as those employed in the life-experiments given by M. Foussat in the 'Electrician' for January 31st.

Mr. Robertson's results from one such lamp are as follow:—

TABLE I.

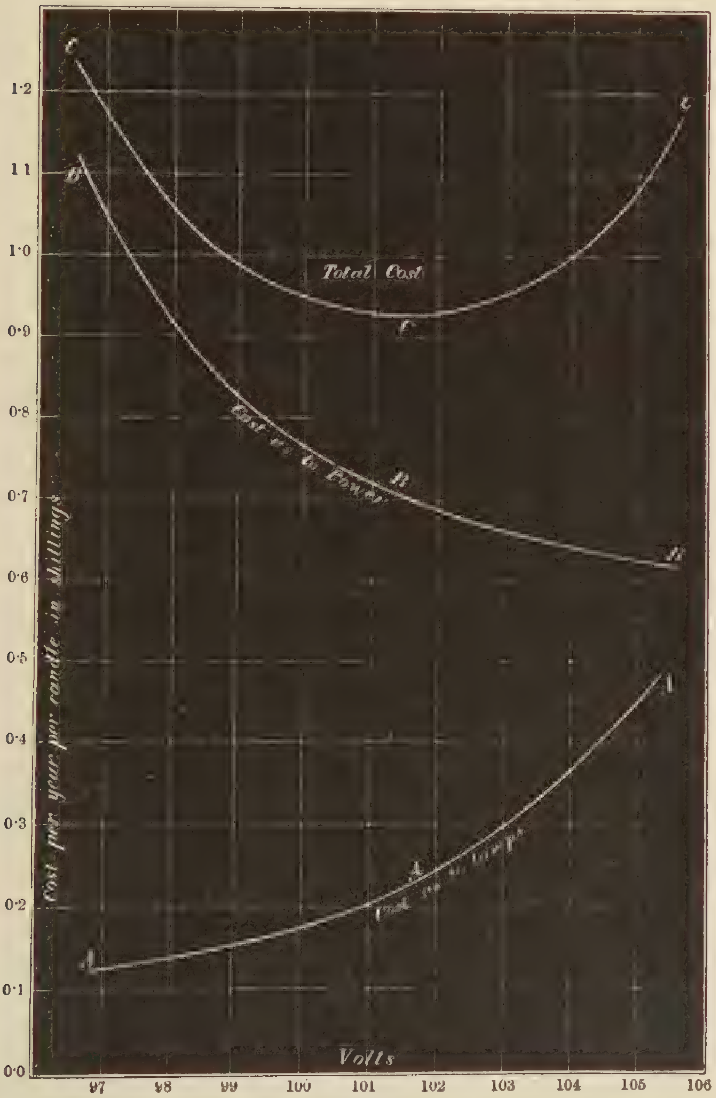
Candles, or $\theta(v)$.	Volts, or v.	Watts.
3	84	56·62
4	87	63·51
5	87·8	66·03
6	90·5	69·05
7	93·5	75·74
10	96·75	79·34
11	97·61	81·01
12	98·04	82·35
14	98·9	85·55
16	100·6	90·54
18	101·5	92·37
20	103·6	96·66
22	105·4	100·10
24	107·5	104·28
26	109·2	108·11
28	110	111·10
30	111·4	113·63
35	114	118·56
40	115·2	122·11

We are therefore now in a position to take up the problem in a more direct manner, without making any assumption beyond this—that Mr. Robertson's 16-candle 100-volt French Edison lamp is of the same kind as M. Foussat's 16-candle 100-volt French Edison lamp.

Solving the problem graphically, in the way previously described, and using the same values of *p*, *n*, and H—viz. cost of a lamp 5*s*., number of hours of burning per year 560, and annual cost of an electric horse-power for those 560 hours £5,—we obtain the curve A A A to represent the cost per candle per year as regards renewal of lamps, B B B the cost per candle per year as regards power, and the resultant curve C C C as the total cost per candle per year.

The minimum value of this cost appears from the curve C C C to be at about **101·4** *volts, and to equal about* **11d.** *per candle per year.* If the potential-difference be maintained constantly down at 98·7 volts, or up at 104 volts, then the cost becomes 1*s*. per candle per year. It will also be seen from the curves that the yearly cost for renewals of lamps is but a small fraction of the total yearly cost, as long as we are

Diagram for a 16-candle 100-volt Edison Lamp.



using potential-differences of not more than about 100 volts; so that when using, with these lamps, this or a lower potential-difference, the actual sum paid for the lamps is not so very serious an item in the yearly lighting bill. But if, on the other hand, we maintain a potential-difference of, say, 104 volts constantly at the terminals, then the lamp renewals represent, in our particular case, nearly one third of the yearly expenditure, and consequently any change in the price of lamps becomes extremely important.

We find that the simplest expression for the candle-power of this incandescent lamp in terms of volts is

$$\theta(v)=a(v-b)^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Thus, for example, it will be found that if the cube roots of the values of $\theta(v)$, given in the preceding table, are plotted with the corresponding values of v as coordinates of points on squared paper, the points lie in no regular curve so nearly as in a straight line, discrepancies being apparently due to errors of observation, and these errors of observation follow a periodic law which may be of interest physiologically. Or it may be that there really is a point of inflexion in the curve.

For the particular lamp in question we find that

$$a=0{\cdot}0002621,$$
$$b=62{\cdot}12\,;$$

or, if the law remains true for a less number of volts than 84, the lowest used in this experiment, then when $v=62{\cdot}12$, the candle-power is zero.

We do not, however, find that the cube roots of the candle-powers of all sorts of incandescent lamps follow a line-function of the difference of potential at their terminals; for on examining the results of experiments that have been made with various types of lamps, we find that with the first five of the following set of lamps the law

$$\sqrt[3]{\theta(v)} \propto v-b$$

is very nearly true.

1. A Lane-Fox lamp	from	50 to 80 volts.
2. A British Electric Light Co.'s lamp	,,	50 to 68 ,,
3. An Edison "7 B" 8-candle lamp .	,,	45 to 63 ,,
4. An old Swan lamp	,,	40 to 64 ,,
5. A low-resistance 8-candle Swan lamp	,,	16 to 40 ,,

Whereas with the next five lamps it is not so true, although for half the range the law might be regarded as true in all the cases.

6. A Maxim lamp . . . . from 42 to 57 volts.
7. A Maxim lamp . . . . „ 35 to 110 „
8. An old form Swan lamp . . . . . „ 25 to 54 „
9. A high-resistance Swan lamp . . „ 45 to 90 „
10. An Edison B 8-candle lamp . . „ 36 to 85 „

We may here mention that we have found the following plan very useful in helping us to draw correctly a curve at a place where there is a sudden bend and an absence of points, determined from experiments to guide us in drawing the curve. Instead of plotting x and y, and obtaining a curve which it would be very difficult to draw correctly, we may by using some simple function of y obtain points which obviously lie in a curve which it is easy to draw. Thus we may plot $\sqrt{y}$ and x, or $\sqrt[3]{y}$ and x, or $\log y$ and x. When the curve connecting y and x is not roughly asymptotic to the axis of x, but to some line parallel to this axis, it is obvious that there is a greater likelihood of obtaining simple curves by plotting $\sqrt{y \pm \alpha}$ and x, or $\sqrt[3]{y \pm \alpha}$ and x, or $\log (y \pm \alpha)$ and x, where α is some constant obtainable by inspection, than by simply plotting the function of y alone.

The following table gives M. Foussat's lives in terms of v and the corresponding values which we have calculated of $\log f(v)$, $\log \theta(v)$, and of $\log f(v)\theta(v)$ from the values of $\theta(v)$ given in the previous table.

Table II.

v.	$f(v)$.	$\log f(v)$.	$\log \theta(v)$.	$\log f(v)\theta(v)$.
95	3595	3·5557	0·9729	4·5286
96	2751	3·4395	1·0119	4·4514
97	2135	3·3294	1·0485	4·3779
98	1645	3·2161	1·0851	4·3012
99	1277	3·1062	1·1199	4·2261
100	1000	3·0000	1·1556	4·1556
101	785	2·8949	1·1898	4·0847
102	601	2·7789	1·2216	4·0005
103	477	2·6785	1·2531	3·9316
104	375	2·5740	1·2852	3·8592
105	284	2·4533	1·3152	3·7685

When $\log f(v)\, \theta(v)$, and v are plotted as coordinates of points, these points are found to lie so nearly on a straight line that the formula

$$\frac{1}{f(v)\theta(v)} = 10^{0·07545\,v-11·697} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is found to be true with considerable accuracy, and such a formula lends itself with great ease to calculation.

It is quite true that we might have obtained a still more accurate formula than (2) since, as published by Mr. Wright, one of the members of our class, in the 'Electrician' for February 21st, the logarithm of the life of a lamp is shown to be a line-function of the difference of potentials at which it is worked.

We may in fact with this particular type of lamp put

$$f(v) = 10^{14-0·11\,v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

with very great accuracy indeed.

Combining this with (1), we obtain as a more accurate formula,

$$\frac{1}{f(v)\theta(v)} = \frac{1}{a}\,10^{0·11\,v-14}(v-b)^{-3}. \quad . \quad . \quad . \quad (4)$$

But unfortunately this more accurate formula does not lend itself to mathematical calculation, whereas that given in (2) is very suitable for this, and has a sufficient degree of accuracy for our purpose. In using (2) we are really using for the candle-power

$$\theta(v) = 10^{0·03455\,v-2·303} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

instead of (1).

It will be also found that from 95 to 105 volts, the range of volts given in Table II., the values of $\phi(v)$, or watts per candle-power, when corrected for errors of observation, satisfy with considerable accuracy the equation

$$\phi(v) = 3·7 + 10^{8·007-0·07667\,v}. \quad . \quad . \quad . \quad . \quad (6)$$

If, however, we take the whole range of values given in Table I., then it will be found that the equation

$$\phi(v) = 2 + 10^{4·424-0·03793\,v} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

is better satisfied than (6).

The following Table III. gives the numbers we have actually employed in making these calculations.

TABLE III.

v.	$\sqrt[3]{\theta(v)}$. corrected.	$\theta(v)$ corrected.	Watts corrected.	Watts per candle-power $\phi(v)$.	$\log\{\phi(v)-2\}$.
84	1·4	2·744	56	20·408	1·2650
86	1·53	3·582	60·15	16·79	1·1700
88	1·66	4·574	64·25	14·27	1·0888
90	1·786	5·7	68·4	12·0	1·0000
92	1·914	7·0	72·6	10·37	·9227
94	2·044	8·55	76·85	9·0	·8451
96	2·175	10·28	81	7·882	·7695
98	2·3	12·167	85·15	7·0	·6990
100	2·428	14·24	89·3	6·271	·6305
102	2·556	16·69	93·5	5·6	·5563
104	2·682	19·29	97·8	5·07	·4871
106	2·81	22·188	101·9	4·593	·4138
108	2·94	25·412	106	4·172	·3369
110	3·065	28·79	110·1	3·824	·2610
112	3·195	32·61	114·25	3·504	·1772
114	3·32	36·594	118·3	3·233	·0910

Using (6) and (2), the total cost of one candle per year is

$$p\,n\,10^{0{\cdot}07545\,v-11{\cdot}697}+\frac{\mathrm{H}}{746}\left(3{\cdot}7+10^{8{\cdot}007-0{\cdot}07667\,v}\right).$$

This is a minimum when

$$v=110{\cdot}66+6{\cdot}574\log\frac{\mathrm{H}}{p\,n}.$$

Hence taking $p=0{\cdot}25$, $n=560$, H = £5, we find

$$\mathbf{v}=\mathbf{101{\cdot}15}\text{ volts}.$$

Using (7) and (2), we should find in the same way that the minimum is obtained when

$$\mathbf{v}=\mathbf{101{\cdot}46}\text{ volts};$$

and these minimum values of v agree very closely with that previously determined graphically.

One very important problem in connection with incandescent lamps, and one that cannot yet be regarded as solved, is the determination of the life of a lamp for any given number of volts, from experiments made either on the efficiencies at several different potentials, or from experiments on the life made at so high a potential-difference that the life will be short, and the experiment made therefore in a comparatively short time.

If, however, an expression of the form

$$f(v)=10^{a-bv}$$

can be regarded as representing with sufficient accuracy the law of life for all types of incandescent lamps, then if sufficient experiments be made with a number of lamps at each of two different potential-differences to enable us to determine the average life at each of these potential-differences, the constants a and b can be calculated from the equation, and hence the value of $f(v)$, the life, calculated for any other potential-difference.

In connection with this investigation we have endeavoured, with the aid of one of our Assistants, Mr. Walmsley, to ascertain whether some form of ordinary cheap candle could be used, at any rate for rough photometric measurements, in place of the much dearer standard-candles, and, as far as the following results obtained with seven candles selected at random from a packet of No. 8 sperm-candles go, it would seem that these candles do not differ so very much more in intensity from one another than standard candles are said to do. Of course many more experiments on this subject must be made before the possibility of using cheap candles as a rough standard can be decided on, but in the meantime the following experiments may be interesting.

This particular type of candle, No. 8 sperm, and costing 11*d.* per pound, was selected, because such candles were found to resemble in thickness the standard candles that we have been accustomed to use, and which cost 2*s.* 9*d.* per pound.

TABLE IV.

Name of candle.	Grammes of wax burnt per hour.	Light in terms of that emitted by the Standard candle.
Standard candle ...	7·82	1·00
1st Sperm " ...	Doubtful	1·14 (?)
2nd " " ...	6·102	1·00
3rd " " ...	7·188	1·00
4th " " ...	7·29	1·02
5th " " ...	7·1	1·02
6th " " ...	6·84	1·05
7th " " ...	6·66	0·99

These tests of candle-power were not made with any very high degree of accuracy; but the comparison of these No. 8 sperm-candles with the standard was carried out with probably quite as much accuracy as is employed in making ordinary commercial experiments on the luminosity of incandescent lamps.

VII. *On the Second Law of Thermodynamics.*

By Mr. J. Macfarlane Gray*.

Mr. Macfarlane Gray gave an account of a most extended investigation upon the second law of thermodynamics. From considerations connected with the specific heats of liquids and gases, the author comes to the conclusion that the second law is not true. The experimental results used are chiefly those of Regnault, to which, however, Mr. Gray has applied some corrections.

VIII. *Lecture-Experiments on Spectrum Analysis.*

By E. Cleminshaw, *M.A., F.C.S.*†

I propose in the following paper to explain some processes by which all the phenomena in spectrum analysis, usually shown upon the screen in class-demonstrations, may be exhibited without the use of the electric light, which for various reasons is frequently not available. The methods employed depend upon the use of the limelight and the oxy-hydrogen flame: these of course do not give the same brilliant effects as the electric light, but all the phenomena may be shown upon the screen with much simpler apparatus; and for some of my methods I think I may claim advantages over the usual method of demonstration.

i. *Spectra of the Alkalis and Alkaline Earths.*

Debray (*Ann. Ch. Phys.* [3] lxv. p. 331) proposed the use of the oxy-hydrogen flame for the volatilization of the alkalis and alkaline earths. By a modification of the ingenious method of Bunsen for procuring a monochromatic flame, the substances in question may be conveniently introduced into the flame. Bunsen made hydrogen in a bottle containing a strong solution of sodium chloride with zinc and hydrogen sulphate, passed coal-gas into the bottle in order to increase the size of the flame, and to assist in the mechanical carrying-over of the spirtings of the sodium-chloride solution, and burnt the mixed gases in a description of Bunsen burner.

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* Read February 28, 1885.

† Read March 14, 1885.

Instead of burning the mixed gases in air, I propose to burn them in an atmosphere of oxygen, by which I find a sufficiently bright flame can be obtained to show the spectra on the screen.

The mixed gases are burnt from a jet consisting of two concentric brass tubes, the inner of which passes into the gas-generating apparatus, and the outer is connected with a supply of oxygen; the inner tube is from $\frac{1}{8}$–$\frac{3}{16}$ inch diameter, and may with advantage be wider at the bottom. A better result is obtained by passing hydrogen into the bottle, since it is not easy to render the coal-gas perfectly non-luminous by the oxygen.

A rapid evolution of hydrogen is necessary for a good result. The lens or lenses should be arranged so as to obtain as bright an image of the slit as possible, even at the expense of some spherical aberration.

The light obtained with sodium chloride is perfectly monochromatic and very bright. I venture to recommend this method of obtaining a bright monochromatic flame for lecture-demonstrations.

With a strong solution of lithium chloride the red band (Li *a*) can be shown, and on adding some strong solution of sodium chloride the yellow band of sodium.

For showing the spectra of the alkaline earths I use a saturated solution of the chlorides and generate hydrogen with zinc and hydrogen chloride. With two CS_2-prism bottles a red, orange (Sr *a*), and blue (Sr δ) can be shown on the screen; and, if the slit is narrow, the yellow band of sodium is clearly shown as soon as some sodium-chloride solution is introduced into the gas-generator. If a strong solution of lithium, sodium, and calcium chlorides are used, with two CS_2-prisms five bands can be shown on the screen, including the orange, green, and violet bands of Ca.

The above method is applicable for all compounds which can be volatilized in the oxy-hydrogen flame, and which are not reduced by the nascent hydrogen.

If hydrogen gas is passed into the bottle, or sufficient oxygen used to destroy the luminosity of the coal-gas, no trace of a continuous spectrum can be seen An occasional addition of acid, to maintain the evolution of hydrogen, keeps the flame burning brilliantly for some time ; this method might, therefore, be useful for spectroscopic observations for comparison.

ii. *Continuous Spectrum showing Bright Bands.*

If the slit is narrow, the orange and green bands of calcium may often be seen on the spectrum obtained with the limelight. If the chlorides of the alkalis or alkaline earths are previously melted upon the surface of the lime-cylinder, the bands of these compounds become very bright. Four different compounds might be melted upon different portions, and the spectra of each shown in turn by turning the cylinder. The lime-cylinder should be a "hardlime;" the softer cylinders are too porous.

iii. *Reversed Spectra.*

The important fact that the light emitted by incandescent sodium vapour is opaque to light from the same source, is easily shown in a striking manner upon the screen by placing the apparatus used for showing the yellow band of sodium in § i. in the centre of an ordinary optical lantern ; a bright disk of monochromatic sodium-light may be obtained on the screen from 4–5 feet in diameter. A small Bunsen burner is then placed as near as possible to the condensing lens, with the top of the tube projecting ½ inch above the edge of the lens, which shows the shadow of the tube at the top of the disk upon the screen. When a platinum wire containing a quantity of sodium chloride is introduced into the flame of the Bunsen burner, the dark appearance of the cooler flame is clearly seen on the screen.

Reversal of D *line.*

Mr. Lewis Wright states that this can be done with the limelight by burning sodium in a Bunsen burner placed as close to the slit as possible. I have not been able to obtain very satisfactory results by this method; the flame is too hot. I have been more successful with the following methods:—The sodium is burnt in a small spoon placed between the lens and prism, *just* below the focus of the rays from the lens, which must be made as small as possible, so that all the rays are made to pass through the flame where the sodium vapour is densest, and none of the light from the sodium vapour is focused on the screen, and a much smaller flame may be used. Or the spoon may be placed a little in front of or beyond the focus in the path of the rays; a shadow of the spoon is seen across the spectrum, and with a small flame a marked thickening of the dark line is seen just above the spoon. The sodium may be burnt in a Bunsen burner or in the flame of a spirit-lamp.

If the sodium-flame is placed between the slit and the lime-light, I have obtained a good result by using a Bunsen burner, the flame of which is cooled down to a proper temperature by a mixture of air and carbon dioxide. Care must be taken not to pass carbon dioxide into the flame to excess, otherwise too great a lowering of its temperature takes place. I have, for instance, obtained in this manner a bluish flame which did not show the slightest trace of sodium in a room where sufficient sodium had been burnt to make every gas-flame give a strong sodium reaction.

The supply of carbon dioxide to the Bunsen burner may be adjusted in the following manner:—A cork is attached to the movable cover which closes the two holes for admitting air, and two holes made in it opposite the air-holes; to one a glass tube is attached which is connected with a bottle, into which carbon dioxide is passed, fitted with three openings. By opening clamps the carbon dioxide may be all passed into the Bunsen burner, or passed directly out from the bottle. The sodium is first brought into vivid combustion, and then the flame cooled down by admission of the proper supply of carbon dioxide.

The reversed line may also be shown by burning sodium in a spirit-lamp with four wicks, in the centre of which is a jet for the admission of oxygen. This is placed between the slit and the limelight. On passing oxygeninto the flame, the heat may be raised sufficiently high to produce a bright

Na band upon the screen, especially if the light from the incandescent lime is somewhat moderated, and turned into a dark band when the oxygen is shut off, proving that the production of a dark or bright sodium-band depends upon the temperature of the absorbent vapour.

IX. *On the Characteristic Curves and Surfaces of Incandescence Lamps.* *By* J. A. Fleming, *M.A., D.Sc.* (*Lond.*), *Fellow of St. John's College, Cambridge*[*].

Recent issues of a scientific journal[†] have contained some interesting letters and notes on the life of incandescence lamps, and on resulting deductions to be made therefrom. The difficulty of the full and complete discussion of this subject is the absence of sufficiently prolonged experiments to give statistics reliable for this purpose. These can only be obtained at great expense and by experiments lasting over a considerable time; but the results to hand make a preliminary investigation interesting on the connection which exists between the life of incandescence lamps and other correlated quantities.

The manufacture of incandescence lamps has now advanced to such a condition that the accidents of manufacture are greatly under control. The conditions necessary to get a good lamp are fairly well understood, and the physical actions going on in the lamp are also to a great extent known. We now know that the expectations of earlier investigators of getting an absolutely unalterable carbon incandescence lamp are not destined to be fulfilled; but we know that the gradual destruction of the filament is an operation dependent upon several causes, which may be greatly delayed by attention to, and success in, certain operations of manufacture.

The gradual destruction of the carbon filament in a vacuum lamp is a kind of erosion taking place at one or more points. Observation seems to show that in a single loop-filament this cutting through takes place most generally near the negative side. The determining cause of breakage is, however, temperature ; and carbon filaments which present such inequalities of resistance as to give rise to spots of higher temperature might, other things being equal, be expected to be doomed to a short career. Lamp-filaments are therefore like human lives : some come into the world with a taint of disease upon them, in the shape of irregularity of structure, which predisposes to an early death ; but nevertheless, as even in the case of suicide, the great law of averages overrides particular instances, and gives us, in the case of a sufficiently extended series of observations, a law connecting the average behaviour under fixed circumstances.

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[*] Read March 14, 1885.

[†] 'The Electrician,' vol. xiv. (1885), pp. 246, 294, 311, 347.

Experience gained during three years of commercial manufacture and use of incandescence lamps has demonstrated that when a large number of filaments are prepared with identical care, and the lamps made with them sorted out into batches, and worked with varying electromotive force, there is a very constant relation between the average efficiency or candles per horse-power and the average duration or life, and the working potential or volts of the lamp. Now, in a general way we do not know what the form of the function is that connects these three quantities, or any two of them, but there have been a large number of observations on the relation of certain variables which indicate as a most probable form an exponential function.

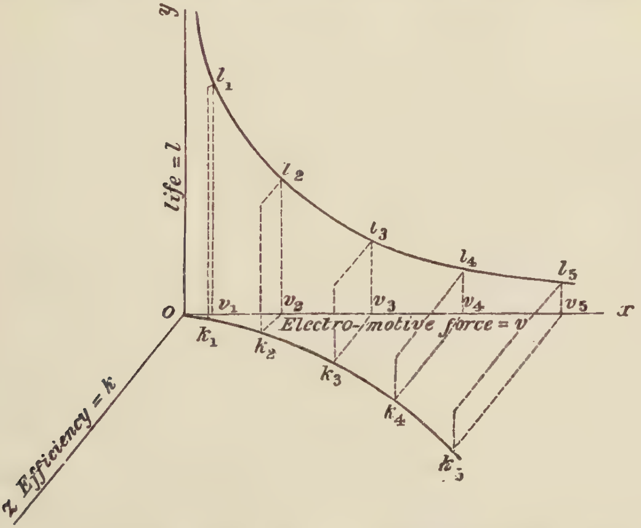
There are four variables between which a relation is required —electromotive force, resistance, candle-power, and life. The third of these is at present somewhat vague and indeterminate. What we really are concerned with is the total eye-affecting radiation ; and our present methods only allow of a certain more or less imperfect comparison of this as a whole with that of a standard candle, or a calculation of the integral deduced from observation of the relative intensities of certain rays. Accordingly, of these four quantities two alone can be measured with any great accuracy. One is merely an average, and the other is necessarily a somewhat ill-defined quantity.

Between any two of these variables we can seek a relation, and having a number of observations we plot down these in what are best called the characteristic curves of the lamp. Now two of the most important of these curves are those connecting the electromotive force and efficiency, or candles per horse-power, and the electromotive force and life; and they may therefore be called the principal characteristic curves of the lamp. Three other useful curves may be obtained by plotting down the curves connecting candle-power and current, candle-power and electromotive force, and electromotive force and resistance. These may be called subsidiary characteristic curves.

Since the life and efficiency of a lamp vary together with the electromotive force, we can only properly represent the relation between the three by a *surface*, which may be called the characteristic surface of the lamp.

Take three rectangular axes, x, y, z (fig. 1), and let distances measured outwards represent life, candles per horse-power, and electromotive force. Let a curve be drawn on the $y\,x$ plane, representing the relation of electromotive force and life, and one on the $x\,z$ plane representing electromotive force and efficiency; let the ordinates be drawn at various points to both these curves, starting from abscissæ, representing certain pressures v_1, v_2, &c. Complete the rectangles on the ordinates k, l, &c.; and we see that these rectangles form the

Fig. 1. Diagram of the principal Characteristic Curves of a Lamp.



orthogonal sections of a solid bounded respectively by the two planes $x y$ and $x z$, and two curved surfaces, of which the characteristic curves l and k are the traces on these planes. The surface of this volume may be called the characteristic surface of the lamp. We see that the area of the orthogonal section parallel to $y z$ gradually increases to a maximum, and then decreases. This is obviously because, for zero electromotive force, life is infinite and candles per horse-power zero; whilst for very high electromotive force, life is zero and candles per horse-power or efficiency a maximum. Now the area of cross section which is a maximum for a certain value v_3 of electromotive force represents the product of life and candles per horse-power, or the maximum candle-hours per horse-power it is possible to get; and the value of $k l$ is therefore a very important quantity. I shall call this maximum value of $k l$ the principal modulus of the lamp, because the value of the lamp for commercial purposes is obtained by dividing the numeric representing this principal modulus by the price of the lamp, taking either the cost of manufacture or the selling-price, according as the question is considered from a manufacturer's or purchaser's point of view.

The product $k l$ is itself a function of the electromotive force, and the value of the electromotive force which makes this quantity a maximum is an important one to determine. It is the pressure at which the lamp should be worked in order to realize the greatest quantity of light for a given expenditure of energy in the lamp.

Let us next consider the form of the function which expresses these characteristic curves. Take, for instance, the curve of life and electromotive force. We do not know whether this curve is a continuous curve, whether it is asymptotic to axis of y, or, in fact, how it behaves beyond the limits of the values of electromotive force v, for which lamps are incandescent. We have, however, certain values for l corresponding to values of v not lying far on either side of the ordinary so-called "marked volts" of the lamp. When the writer was in America he was shown a large number of observations, made with the view of determining the relation between average life and electromotive force, not far on either side of 100 volts, taken for the Edison 108-volt A lamp. For this lamp the life-pressure curve at or near the working-pressure is approximately a logarithmic curve whose equation is

$$l = Av^{-\alpha},$$

where A and α are constants; and for the Edison 16-candle 105-volt lamp α is nearly 25; and, accordingly, life varies inversely as the 25th power, roughly, of the electromotive force. In the figures shown to the writer the life-pressure curve had been drawn for pressures corresponding to lives far in excess of what could actually have been observed during the time filament-lamps have been made, reaching up to 11,793 hours. This is obviously improper: we do not know that α is not itself a function of v, and for pressures departing far from the ordinary working-pressure the variation of average life and pressure is not probably expressed by so simple a relation; and we cannot go fairly beyond the limits of actually observed lives. M. Foussat has recently communicated to a scientific journal the results of observations on the life of French Edison lamps.

TABLE I.

Volts.	Life.	$\log_{10}$ (volts).	$\log_{10}$ (life).
95	3595	1·97772	3·55570
96	2751	1·98227	3·43949
97	2135	1·98677	3·32940
98	1645	1·99123	3·21617
99	1277	1·99564	3·10619
100	1000	2·00000	3·00000
101	785	2·00432	2·89487
102	601	2·00860	2·77887
103	477	2·01284	2·67852
104	375	2·01703	2·57403
105	284	2·02119	2·45332

Now the figures given by M. G. Foussat on p. 246 of 'The Electrician' for 1885, stated to be the result of a large number of observations on the relation of average life and pressure at or near 100 volts, conform approximately to the above law. Taking M. Foussat's numbers for life and electromotive force, and taking logarithms of both, we have the figures given in Table I.

Writing down the differences between successive values of $\log v$ and $\log l$, we get the numbers—

455	11621	432	10513
450	11009	428	11600
446	11323	424	10035
441	10998	419	10449
436	10619	416	12071

Now, if $l = Av^{-\alpha}$, then

$$\log l - \log l_1 = -\alpha(\log Av - \log Av_1),$$

where l and l_1, v and v_1 are adjacent values of l and v; accordingly,

$$\alpha = -\frac{\log l - \log l_1}{\log Av - \log Av_1}.$$

The mean of the differences of $\log l$ divided by the mean of the differences of $\log v$ gives 25·4 nearly; and, accordingly, M. Foussat's numbers agree with those found in America in assigning to α a value not far from 25 for the Edison lamp. In the neighbourhood of 100 volts

$$l = Av^{-25};$$

or, average life varies inversely as the 25th power of electromotive force.

This, however, is easily seen to be a very rough approximation. The successive quotients of life-difference by volt-difference are not constant; and the above simple exponential formula cannot be admitted as anything more than a very imperfect connection. In examining the Carlisle Tables of Mortality, which give the expectation of life at every age drawn from a very large number of observations, it was apparent that an empirical formula connecting the two quantities could be obtained of a form

$$\log e = a + bx + cx^2 + \&c.^*,$$

where e = expectation of life at any age x.

I was led therefore to try if such an empirical formula

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[*] Let x be the age, and e the expectation of life at that age. Then by the Carlisle Tables of Mortality at the several ages 10, 20, 30, 40, 50, 60, 70, 80, 90, the corresponding expectations of life are:—48·82, 41·46, 34·34,

better suited the present case; and a process of trial and failure showed that

$$10 \log l = 135 - v - \frac{v^2}{2000}$$

is a formula which gives very nearly correct results in calculating the life of a lamp l, given the working-pressure v in volts. This may be expressed otherwise:—

$$\log l = 13{\cdot}5 - \frac{v}{10} - \frac{v^2}{20{,}000},$$

or

$$l = 10^{13{\cdot}5 - {\cdot}1v - {\cdot}00005v^2}.$$

Calculating by this formula, we get the following values for log l and l, l being the average life in hours of the 100-volt Edison lamp as made in France:—

TABLE II.

Calculated by formula.		Observed.	
l.	log l.	log l.	l.
3539	3·5488	3·5557	3595
2749	3·4392	3·4395	2751
2136	3·3296	3·3294	2135
1658	3·2196	3·2167	1645
1289	3·1100	3·1062	1277
1000	3·0000	3·0000	1000
776	2·8900	2·8948	785
602	2·7798	2·7788	601
467	2·6698	2·6785	477
362	2·5592	2·5740	375
281	2·4488	2·4533	284

It is evident, then, that the simple exponential function does not give nearly so good results as a formula of this latter description; and there is no doubt but that by a suitable selection of constants a formula can be obtained expressing the life of the lamp as a function of electromotive force throughout an observed range, which shall be closely in

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27·61, 21·11, 14·34, 9·18, 5·51, 3·28; and it can easily be found that

$$10 \log e = 17{\cdot}2 - \frac{x}{100} - \frac{14}{10{,}000}x^2, \text{ very nearly;}$$

the calculated values from which are:—49·6, 44, 36·6, 28·6, 20·9, 14·3, 9·64, 5·54, 3·14.

accordance with observed facts*. Until, however, a much larger collection of statistics is obtained we shall not be in any position to determine if these constants are definite for each type of lamp or kind of carbon, and whether the average life of a lamp can be predicted from a knowledge of these constants.

An attempt was made, in the next place, to endeavour to obtain an approximate empirical formula connecting the efficiency of a lamp, or the candles per horse-power, and the electromotive force. On April 13, 1882, Prof. A. Jamieson read a paper before the Society of Telegraph Engineers and of Electricians (Journ. Soc. Tel. Eng. vol. xi. p. 164), "On Tests of Incandescent Lamps," and he has there given a number of tables and curves for different lamps, giving the efficiencies and resistances for various electromotive forces.

These observations afford a convenient means of putting to the test empirical formulæ, because the observations seem to have been carried out with very great care, and being done with secondary and primary batteries as current generators, the observations are more likely to be accurate than when a current from a dynamo machine is used; also because the observations for candle-power were entrusted to Dr. Wallace, gas-analyst for Glasgow, and were therefore in the hands of an observer whose eye was probably more trained to detect minute differences of illumination than one not so familiar with such work.

Professor Jamieson gives one complete table of the constants of an Edison 8-candle lamp over a great range of candle-power. Selecting that portion of the table in which the electromotive force was high enough to illuminate the lamp, we have as follows:—

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* Mr. F. M. Wright has given, in 'The Electrician,' p. 311 (1885), a formula—

$$100 - V = 9{\cdot}098213 \log \frac{L}{1000}.$$

This is equivalent to

$$9{\cdot}09 \log L = 127 - V;$$

and Professors Ayrton and Perry have given

$$L = 10^{14 - {\cdot}11v}$$

or

$$10 \log L = 140 - 1{\cdot}1\, v,$$

both of which are nearly equivalent to the formula given in the text.

TABLE III.

Tests of an Edison 8-candle Lamp, made by Prof. Jamieson, March 8, 1882.

Resistance, R.	E.M.F. in volts, V.	Current, in amperes, A.	Candle-power, K.
63·7	45·9	0·722	5·2
63·3	46·7	0·737	6·2
62·7	48·3	0·77	8·2
61	51·9	0·85	12·9
60·6	53	0·874	14·3
59·3	56·2	0·948	21·3
58·4	58	0·995	25·3
57·8	61·1	1·06	35·8
57	63·1	1·21	43·8

If we take the logarithms of these numbers we have the following table :—

Edison 8-candle lamp.

log R.	log V.	log A.	log K.
1·80414	1·66181	$\bar{1}$·85854	0·71600
1·80140	1·66932	$\bar{1}$·86747	0·79239
1·79729	1·68395	$\bar{1}$·88649	0·91381
1·78533	1·71517	$\bar{1}$·92942	1·11059
1·78247	1·72428	$\bar{1}$·94151	1·15534
1·77305	1·74974	$\bar{1}$·97681	1·32838
1·76641	1·76343	$\bar{1}$·99782	1·40312
1·76193	1·78604	0·02531	1·55388
1·75587	1·80003	0·08279	1·64147

Let E be the watts of the lamp and k the efficiency or candles per horse-power. Then

$$k = \frac{\text{K}\,746}{\text{AV}};$$

and if we calculate the logarithms of the efficiency corresponding to each electromotive-force value, and compare these with four times the value of the corresponding electromotive force, we have the following table :—

log k.	4 log v.	4 log v − log k.	
2·06839	6·64724	4·57885	
2·13834	6·67728	4·53894	Mean =4·52968 =log 33860
2·21611	6·73580	4·51969	
2·33879	6·86068	4·52189	
2·36229	6·89712	4·53483	
2·47457	6·99896	4·52439	
2·51461	7·05372	4·53911	
2·61527	7·14416	4·52889	
2·63137	7·20012	4·56885	

Excepting the first and last values, which are the result of observations on the candle-power at extreme values, the intermediate figures are not very far from constant, and indicate, as a first rough approximation, that efficiency varies as the fourth power of the electromotive force.

At both high and low candle-powers the comparison of the light with a standard candle is difficult. In one case an excess of red, and in the other an excess of violet rays makes the comparison of the naked lights much more difficult at extremes, and, as is well known, the efficiency for very high or low candle-power must be stated for a definite radiation. The only other comparison between these two variables attempted has been in the case of the experiments made by the Committee appointed to report on the incandescence lamps in the Paris Electrical Exhibition, an abstract of which appears in the same volume of the 'Journal of the Society of Telegraph Engineers.' The Committee condense these results on four varieties of lamps into the following numbers for the efficiencies, or candles per horse-power and electromotive force working them, taken at two very different values.

	Edison.		Swan.	
E.M.F.	89·11	98·39	47·3	54·21
Candles per horse-power...	196·4	307·2	177·9	262·5
	Lane-Fox.		Maxim.	
E.M.F.	43·63	48·22	56·49	62·27
Candles per horse-power...	173·6	276·9	151·3	239·4

If we take the logarithm of each number, and compare the difference of the logarithm of the efficiencies with the difference of the logarithm of the corresponding electromotive forces for each lamp, we have the following ratios :—

Edison.

$$\frac{19428}{4302}=4{\cdot}5,$$

Swan.

$$\frac{16895}{5922}=2{\cdot}9.$$

Lane-Fox.

$$\frac{20277}{4344}=4{\cdot}6,$$

Maxim.

$$\frac{19928}{4231}=4{\cdot}7.$$

If, therefore, k_1 and k_2 be the efficiencies corresponding to two observed values v_1 and v_2 of electromotive force,

$$\frac{\log k_1-\log k_2}{\log v_1-\log v_2}=4{\cdot}5\ ;$$

or

$$k=\mathrm{C}v^{4\cdot5}$$

represents an approximate formula for calculating the efficiency. If, now, approximately, in the case of an Edison lamp, the candles per horse-power vary as the 4·5 power of the electromotive force, and the life varies inversely as the 25th power of the electromotive force, it follows that

$$l \propto \frac{1}{k^{5\cdot7}}$$

or, roughly, that the life varies inversely as some power between 5 and 6 of the candles per horse-power.

The writer was given in America an extensive series of observations on the relation between the candles per horse-power and the average life, which were as follows :—

Life.	Candles per H. P.
11793	100
7150	110
4528	120
2974	130
2015	140
1403	150
1000	160
727	170
539	180
406	190
310	200
236	210
102	220

The length of life, 11,793 hours, given as corresponding to an efficiency of 100 candles per horse-power, is nearly equal to four years of average burning in working hours, and could not have been the result of actual observation; but these numbers agree very closely with the law that

$$l \propto \frac{1}{k^{5\cdot25}},$$

where l represents life in hours and k candles per power. It is most probable that this law has been deduced from observations on the life lying within an observed range and then extended by calculation to efficiencies below those actually observed. In any case these observations are not in great discord with the above deductions made from the numbers furnished by M. Foussat on the connection between life and electromotive force, in conjunction with other observations on the relation of efficiency to working pressure in the case of Edison lamps.

It is, however, far more probable that the connection between the candles per horse-power and the working pressure is expressible by a formula of this kind,

$$\log k = \alpha + \beta v + \gamma v^2 + \&c.,$$

where α, β, γ are known constants; and in this case we should have then both the life-pressure characteristic curve, and the efficiency-pressure characteristic curve expressed by analogous equations:—

$$\log l = a + bv + cv^2 + \&c.,$$

$$\log k = \alpha + \beta v + \gamma v^2 + \&c.$$

We have seen above that such a formula does fit in with observed values for one pair of variables. Further examination of this point is desirable.

Professors Ayrton and Perry have drawn attention [*] to a connection between the candle-power of a lamp and the working potential, and they find that in very many cases the cube root of the candle-power is proportional to the working potential minus a constant. Now, this is equivalent to saying that the candle-power of the lamp is proportional to the cube of the potential measured above a certain point.

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[*] "On the most Economical Potential-Difference to Employ with Incandescent Lamps:" 'The Electrician,' March 7, 1885, p. 348.

I have examined some records of measurements of Edison lamps to put this law to further test, and the results are given below for two 16-candle Edison lamps measured by myself, and one 8-candle lamp measured by Prof. Jamieson.

Edison 16-candle Lamp.—No. 1.

Candle-power=K.	Volts, v.	$\sqrt[3]{K}$.	$v-57{\cdot}17$.	$\frac{v-57{\cdot}17}{\sqrt[3]{K}}$.
16	105·27	2·5726	48·10	18·8
11·5	99·56	2·2572	42·39	18·8
8·25	94·32	2·0206	37·15	18·4
4·8	90·03	1·6869	32·86	19
2·8	85·74	1·4095	28·57	20
2	81·22	1·2599	24·05	19·08

Edison 16-candle Lamp.—No. 2.

K.	v.	$\sqrt[3]{K}$.	$v-50$.	$\frac{v-50}{\sqrt[3]{K}}$.
16	103·84	2·5198	53·84	2·5
13	100·03	2·3513	50	21·3
9·5	97·65	2·118	47·65	22·4
8·25	94·79	2·019	44·79	22·1
7	92·89	1·913	42·89	22·3
5·5	88·12	1·765	38·12	22·2

Edison 8-candle Lamp used by Professor Jamieson.

K.	v.	$v-28{\cdot}7$.	$\sqrt[3]{K}$.	$\frac{v-28{\cdot}7}{\sqrt[3]{K}}$.
5·2	45·9	17·2	1·7324	9·928
6·2	46·7	18	1·8371	9·798
8·2	48·3	19·6	2·0165	9·720
12·9	51·9	23·2	2·3453	9·892
14·3	53	24·3	2·446	9·937
21·3	56·2	27·5	2·772	9·921
25·3	58	29·3	2·936	9·981
35·8	61·1	32·4	3·296	9·831
43·8	63·1	34·4	3·525	9·759

In these cases we see that the cube root of the candle-power is very nearly proportional to the excess of the electromotive force above a certain point. Now, on examining a number of such cases, it appears that this constant is the value of that electromotive force at which the lamp just begins to give signs of incandescence. In the case of the 8-candle lamp above, Professor Jamieson marks in his table against the E.M.F. 28·7 "filament bright red." It will be very interesting if further examination should give confirmation to this surmise; but the above figures seem to indicate that the cube root of the candle-power is proportional to the electromotive force reckoned from the neighbourhood of that pressure at which the filament begins to give out light. If we call this excess pressure the "effective volts," then we can state the rule that the cube root of the candle-power is proportional to the effective volts. We have then, according to the observations of Professors Ayrton and Perry, in a certain number of cases an empirical law of this kind,

$$\sqrt[3]{\mathrm{K}}=a(v-b),$$

in which K is the candle-power measured at a pressure v.

This may be written

$$\tfrac{1}{3}\log \mathrm{K}=\log a+\log (v-b).$$

By ordinary algebraic theory we have

$$\log a=\overline{a-1}-\tfrac{1}{2}\overline{a-1}^2+\tfrac{1}{3}\overline{a-1}^3-\text{ \&c.};$$

∴ if $c=b+1$, we can write $\log (v-b)$ as equal to the series

$$\overline{v-c}-\tfrac{1}{2}\overline{v-c}^2+\tfrac{1}{3}\overline{v-c}^3+\text{ \&c.},$$

and

$$\log \mathrm{K}=\mathrm{A}+\mathrm{B}\overline{v-c}+\mathrm{C}\overline{v-c}^2+\mathrm{D}\overline{v-c}^3+\text{ \&c.}$$

If the supposition above made is correct, that b is a value not far from that pressure at which the lamp becomes incandescent, then k is seen to be an exponential function of the effective volts of a kind similar to that which has been found to reconcile very well the values of observed life and corresponding working electromotive force.

Other observers have before now called attention to the fact, that within a certain range the candle-power of a lamp varies approximately as the sixth power of the current passing*.

In the case of the above-mentioned Edison 16-candle lamp,

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* At the British-Association Meeting at Montreal, Mr. Preece read a note on this subject, confirming previous observations made in 1883, and showing, from observations of his own, Professor Kittler, and Captain Abney, that incandescence varies very nearly as sixth power of current.

No. 1, the following values were obtained, connecting current and candle-power :—

Candle-power, K.	$\sqrt[6]{K}$.	Current in amperes, a.	$\sqrt[6]{K} \div a$.
16	1·5874	·7582	2·09
11·5	1·5024	·7065	2·12
8·25	1·4215	·6720	2·11
4·8	1·2988	·6203	2·09
2·8	1·1872	·5859	2·03
2	1·1225	·5514	2·02

The table shows that, with considerable accuracy, over a range from two to sixteen candles, the incandescence varies as the sixth power of the current.

In those characteristics into which candle-power enters in any way, there is a considerable difficulty in getting results with sufficient accuracy to determine the constants of the characteristic equations. There is, however, one pair of variables, namely electromotive force and resistance, both of which can be measured with very high accuracy over a great range; and some very interesting examples of these pressure-resistance curves are given by Professor Jamieson in his memoir above alluded to.

A very short examination of the way in which an incandescence lamp behaves under increasing electromotive force, shows that the resistance decreases with increase of electromotive force, but that it does not decrease without limit; it tends to a minimum value, beyond which it appears to be constant. This is very strikingly shown for some of the Swan lamps tested by Professor Jamieson. In his paper certain formulæ are given connecting various lamp-constants, and as a first approximation to a pressure-resistance equation is given the following :—

$$\log P = \log E + ar,$$

where P is a constant, $E = $ E.M.F., and $r = $ resistance.

It is not possible that such a formula should represent correctly the relation of resistance to pressure at high pressures, because it in no way expresses the fact that resistance tends to a minimum with increasing electromotive force.

An empirical formula can, however, be obtained which will express this in the following way :—

Let R be the resistance of a lamp measured with any electromotive force, E, at the terminals. Let E_0 represent the electromotive force at which the lamp just becomes incandescent, and let R_0 be the corresponding resistance. Let r be the minimum resistance to which the lamp approximates, as E increases, until the filament breaks. Then

$$R = r + 10^{\log(R_0 - r) - A(E - E_0)}$$

is an expression which has the following properties:—

$$\text{If } E = E_0, \quad \text{then } R = R_0;$$
$$\text{if } E = \propto, \quad ,, \quad R = r.$$

The above formula is otherwise written

$$\log(R - r) = \log(R_0 - r) - A(E - E_0).$$

This formula has been tested for an Edison 8-candle lamp of which the resistance was measured over a considerable range by Professor Jamieson, and the resistance-pressure curve given in his paper. For this lamp luminosity commenced at 28·7 volts, and at this pressure it had a resistance of 73·4 ohms; its resistance gradually decreased with increasing pressure until it became apparently constant and equal to 53·5 ohms. Taking A equal to $\frac{1}{50}$, we have

$$50 \log(R - 53{\cdot}5) + (E - 28{\cdot}7) = 50 \log(73{\cdot}4 - 53{\cdot}5) = 64{\cdot}945.$$

Calculating R by this formula for various values of electromotive force E, we have the following table of observed and calculated resistances:—

Edison 8-candle Lamp.

E.M.F.	Resistance observed.	Resistance calculated by above formula.	E.M.F.	Resistance observed.	Resistance calculated by above formula.
28·7	73·4	73·4	48·3	62·7	61·57
32·7	71·1	70·05	51·9	61	60·34
36·1	68·1	67·65	53	60·6	60
39·7	66·4	65·49	56·2	59·3	59·13
43·1	64·7	63·78	58	58·4	58·66
45·9	63·7	62·51	61·1	57·8	57·98
46·7	63·3	62·19	63·1	57	57·58

The accordance between the observed and calculated resistances is fairly close. It seems very probable, from an inspection of the values at very low electromotive forces, that the resistance is a function of the electromotive force, reckoned from the pressure b, at which the lamp-filament begins to be bright red, and that of R is the resistance corresponding to any electromotive force, E, and r is the minimum resistance to which the lamp tends. Then

$$\log (R-r) = A + B(E-b) + C(E-b)^2 + \&c.,$$

where A B C, &c. are constants. If this should be the case, then it may be possible to express the life, candles per horse-power, candle-power, and resistance of a lamp, all as similar functions of the electromotive force, knowing certain constants and the two values of electromotive force at which the lamp becomes incandescent, and the resistance to which it finally tends to obtain. When, however, we are dealing with comparatively small variations of electromotive force, it is possible to calculate both life and efficiency from a simpler exponential function of the form—

$$l \propto v^x$$

$$R \propto v^y$$

when x and y are numerics. Assuming such an approximate formula, it becomes possible to deal with an interesting question, which has been discussed also by Professors Ayrton and Perry*, but which is capable of being investigated in a slightly different manner from that adopted by them in their paper. We shall take the warrant we have in the above figures for the assumption that for electromotive forces not far from those at which the lamp is intended to be used in the case of an Edison lamp, say 100 volts,

Average life varies inversely as the twenty-fifth power of the electromotive force ;

Efficiency, or candles per horse-power, varies as the fourth power of the electromotive force ;

Candle's light varies as the sixth power of the current, and therefore as the sixth power of the electromotive force, seeing that the resistance alters very little after the lamp has reached fair incandescence ;

And that therefore life varies inversely as the sixth-and-a-quarter power of the efficiency, and also inversely as the fourth-and-a-quarter power of the candle-power.

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* "On Potential-difference, &c."

Let p be the price of a lamp in pounds sterling, or fractions of a pound.

Let l be the average life in hours, c the actual candle-power, and k the candles per horse-power, when run at a certain electromotive force v.

Then $\frac{p}{cl}$ is the cost of one candle-light per hour as far as the lamp itself is concerned. Let P be the cost of 1 horse-power hour of electric energy expended in the filament. P will not be the same for all amounts, 1000 P costs less than ten times 100 P, but for the small variations we are considering we shall consider P to be a constant. Then $\frac{\mathrm{P}}{k}$ is the cost of the power in making one candle-light for one hour, and the total cost of getting one candle-light for one hour is

$$\frac{p}{cl}+\frac{\mathrm{P}}{k}=\mathrm{T}.$$

The first term is the expense of the translating device, and it may be called the cost of *lampage.*

The second term is the cost of the production of the energy which passes through the translating device or lamp-filament and is converted into eye-affecting radiant energy, and this may be called the *machinage*; hence the total expense of keeping going incandescent light is made up of lampage and machinage. Now, we can procure a given amount of light either by running the lamps very high, in which case lamps will cost a great deal, and power less in proportion, or we can run the lamps very low and save in lampage, whilst expending more in power in a greater number of lamps; and the question arises, apart from capital expenditure, at what point is the greatest economy obtained, or, in other words, what proportion ought lampage to bear to machinage in order that the total cost may be a minimum? To solve this we shall assume, as is very probable, that for the limit of variation of E.M.F. employed the average life of the lamp is an exponential function either of the candles per horse-power k, or of the candle's light c, for the particular lamps considered.

Let $l=\frac{\mathrm{A}}{k^{\alpha}}$ and $l=\frac{\mathrm{B}}{c^{\beta}}$ be the functions. α and β have definite values at a given E.M.F.; A and B are constants.

Then $cl = B'l^{1-\frac{1}{\beta}}$; $k = A'l^{-\frac{1}{\alpha}}$.

Substituting, we have

$$B'pl^{\frac{1}{\beta}-1} + A'Pl^{+\frac{1}{\alpha}} = T;$$

in which B' and A' are constants.

Now this expresses the total cost of working as a function of the average life at a certain E.M.F.

Let us also take that $l = \frac{1}{v^\gamma}$, where v is the E.M.F. at which the lamps are being run.

Then

$$B'p\left(\frac{1}{v^\gamma}\right)^{\frac{1-\beta}{\beta}} + A'P\left(\frac{1}{v^\gamma}\right)^{\frac{1}{\alpha}} = T,$$

or

$$B'pv^{-\frac{1-\beta}{\beta}\gamma} + A'Pv^{-\frac{\gamma}{\alpha}} = T.$$

Let this be varied by varying v; to find at which value it becomes a minimum with respect to v, differentiate with respect to v, and equate to zero.

$$\frac{dT}{dv} = 0 = -\frac{B'\frac{1-\beta}{\beta}\gamma pv^{-\frac{1-\beta}{\beta}\gamma}}{v} - \frac{A'\frac{\gamma}{\alpha}Pv^{-\frac{\gamma}{\alpha}}}{v},$$

or

$$-\alpha\frac{1-\beta}{\beta}B'pv^{-\frac{1-\beta}{\beta}\gamma} = A'Pv^{-\frac{\gamma}{\alpha}}.$$

Hence the total cost of working is a minimum when the cost of power $A'Pv^{\frac{\gamma}{\alpha}}$ is to the lampage $B'pv^{-\frac{1-\beta}{\beta}\gamma}$ as $\alpha\frac{1+\beta}{\beta}$ to unity.

Now the above investigations show that for Edison lamps α is a quantity in the neighbourhood of $6\frac{1}{4}$, and β near $4\frac{1}{6}$; hence

$$-\alpha\frac{1-\beta}{\beta} = \frac{19}{4},$$

and hence ratio of lampage to total cost is $\frac{4}{23} = 17{\cdot}4$ per cent.

Hence we arrive at this curious result, that independently of the cost of the lamp, or electrical energy, we must run at such a pressure that lampage is about 18 per cent. of the total cost. Now, if instead of employing these approximate exponential expressions for the values of l, c, and k in terms of the electromotive force, we had introduced the more accurate forms of equation indicated above, we should have had an equation in terms of v to solve as the result of equating the differential to zero, which would, by the introduction of the proper constants, give the value of electromotive force at which the total cost becomes a minimum. In their paper, Professors Ayrton and Perry have calculated to a fraction of a volt what this economical potential is. As, however, the characteristic equations are only approximate, it seems hardly necessary to do more than obtain a similar approximate expression for the economical working.

At the Edison lamp-factory in America calculations were made, on the assumption of a particular type of lamp and length of life and cost of power, to ascertain the ratio of lampage to total cost, which made the total cost a minimum, and the result appeared to be to fix it at about 16 per cent. These calculations are therefore in singular accord with the deduction of theory based on determination of the constants of the characteristic curves, arrived at both by graphic and analytical methods.

X. *On the Structure of Mechanical Models illustrating some Properties of the Æther.* *By* Prof. George Francis Fitzgerald, *F.R.S.**

The elements of which the model is constructed consist of pairs of wheels so geared together that when one of them rotates it causes the second to rotate in the same direction. The simplest way of effecting this is to connect them by a band, and this is sufficient for a one-dimensional model. Such a model may be constructed by fixing a number of wheels with their axes parallel and at right angles to a plane, and connecting each wheel with its neighbours by elastic bands. This represents a nonconducting region of the æther. A perfectly conducting region is one in which there are no bands, and a partially conducting region would be represented by the bands slipping more or less. A short description of how electrostatic, electrokinetic, and luminiferous phenomena are illustrated by such a one-dimensional model will be clearer than the corresponding description of the tridimensional model, the structure of which is the special purpose of this paper.

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* Read March 28, 1885.

As an illustration of an electrostatic phenomenon, consider two conducting regions separated by a nonconducting region everywhere except along one line where the bands are removed. If anywhere in this line a rotation in opposite directions be communicated to the wheels that abut on it, then all the wheels in the nonconducting region will be turned more or less. If anywhere two neighbouring wheels turn equally, there is no straining of the band connecting them; but if one turn more than the other, the connecting band is strained and one side becomes tight and the other loose. Now it will be found in the case considered that all the bands are strained, and that all the tight sides are turned towards one conductor and all the loose sides towards the other. This represents the charging of the two conductors in opposite ways. The strain of the bands in any element of the medium represents the polarization of the element, and the line joining the tight and loose sides is the direction of polarization or of electric displacement. The energy of the system is in the form of this straining of the bands, which produces stresses tending to restore the unstrained condition. With a given strain at their respective surfaces, there would be more elements involved and more energy in the medium when the conductors are far apart than when near, showing that if we could represent in any way the fact that conductors can move through the æther there would be forces tending to produce this motion, or, in other words, there would be attraction between these oppositely electrified bodies. As, however, the model does not illustrate the connection between matter and æther, neither this nor magnetic attractions are represented, nor have electromotive forces such as exist in cells been represented. If the forces that have been supposed to turn the wheels along the conducting line connecting the two conducting regions cease to act, the state of strain will disappear, and what represents an electric discharge along this line will take place. All along this line, during the discharge, the wheels at opposite sides will be rotating in opposite directions; so that this is what represents an electric current at any point. It is the same as an electric displacement; and in a nonconductor such opposite rotation is resisted by the stresses in the band, but in a conductor it may take place to any extent. During the discharge the whole of the nonconducting region is full of rotating wheels, and their axes of rotation are at right angles to the direction of discharge. Their velocity of rotation evidently represents the magnetic force accompanying the discharge, and the momentum of the wheels represents the electrokinetic momentum of the current, *i.e.* its self-induction. This is further illustrated by this, that if the frictional resistance be small enough, this momentum will carry the wheels beyond their positions of equilibrium, and there will result an oscillating discharge, such as occurs when an electric condenser is discharged through a sufficiently small resistance. If we suppose a certain amount of frictional resistance at any point along the line of discharge, we may see that the energy expended on friction is conveyed to the place by the bands in the surrounding nonconductor and comes in at the side of the conductor, in accordance with Prof. Poynting's theorem as to the direction of the flow of energy in an electrodynamic system.

The mutual induction of two circuits may also be illustrated by the model. Sufficient has been explained, however, to show how electrostatic and electrokinetic phenomena are represented on the model. If a sudden movement of rotation be communicated to any set of wheels, it is evident that inertia will prevent their neighbours being instantaneously turned, while the connecting bands will be strained. Rotation will, however, be communicated to the neighbouring wheels, and from them to their neighbours, by a process which is a species of wave-propagation. If we consider the nature of the disturbance which is thus propagated, we see that it consists in a rotation whose axis is at right angles to the direction of propagation, and of a polarization of the bands which is at right angles both to the axis of rotation and to the direction of wave-propagation. These are respectively a magnetic and an electric displacement, which are at right angles to one another and to the direction of wave-propagation. This is exactly in accordance with Maxwell's electromagnetic theory of light-propagation. It is thus seen how the same model that can represent electrostatic and electromagnetic phenomena also illustrates luminiferous phenomena by its small oscillations.

If we try to produce a tridimensional model by means of wheels geared together by bands, we are met by the following difficulty. The energy of the model we have been considering may be represented in the following way:—Let ζ represent the angular rotation of any wheel from a given position; then the kinetic energy of an element will evidently be proportional to $\dot{\zeta}^2$, while the potential energy of an element, depending as it does on the difference of rotation of neighbouring wheels, will be proportional to $\left(\frac{d\zeta}{dx}\right)^2+\left(\frac{d\zeta}{dy}\right)^2$. If we were to build up a tridimensional model by simply putting together three such systems of wheels in three rectangular planes, the kinetic energy of an element of the model would be proportional to

$$\xi^2+\eta^2+\zeta^2,$$

but its potential energy would be simply

$$\left.\overline{\frac{d\zeta}{dx}}\right|^2+\left.\overline{\frac{d\zeta}{dy}}\right|^2+\left.\overline{\frac{d\eta}{dz}}\right|^2+\left.\overline{\frac{d\eta}{dx}}\right|^2+\left.\overline{\frac{d\xi}{dy}}\right|^2+\left.\overline{\frac{d\xi}{dz}}\right|^2,$$

instead of

$$\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)^2+\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)^2+\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)^2,$$

which is what it should be if it is to represent the æther.

The electrostatic and electrokinetic energies of an element of the æther may be expressed in this form by assuming ξ, η, and ζ such that their velocities $\dot{\xi}$, $\dot{\eta}$, and $\dot{\zeta}$ are α, β, and γ, the components of the magnetic force at the element. Now simple band-gearing will not enable us to arrange that no straining shall result, when, for instance, $\frac{d\eta}{dx}=\frac{d\xi}{dy}$; but this may be accomplished by the following arrangement:—Each element of the æther is to be represented by a cube, on each edge of which there is a paddle-wheel. Thus on any face of the cube there will be four paddle-wheels. Now, if any opposite pair of paddle-wheels on a face rotate by different amounts, they will tend to pump any liquid in which the whole element is immersed into or out of the cube, and if the sides of the cube be elastic there will be a stress which will tend to stop this differential rotation of the wheels. If, however, the other pair also rotate by different amounts they may undo what the first pair do; and thus the stress will depend on the difference between the differential rotations of these opposite pairs of wheels, *i.e.* on $\frac{d\eta}{dx}-\frac{d\xi}{dy}$.

In order that these four wheels may not similarly work with any other wheel in the cube it is necessary to place diaphragms, cutting the cube into six cells, each a pyramid standing on a face of the cube. These must be so made that liquid may not be able to pass from one cell to another through the diaphragm, nor beside the paddle-wheels. In order actually to effect this, the floats on the paddle-wheels would have to be drawn down while passing the diaphragm (of course these mechanical details could hardly be carried out so as to work with sufficiently little friction for the working of any actual model to approximate sufficiently closely to that of the æther for it to be worth while attempting to construct it). The faces of the cube should be filled up with diaphragms past which the paddle-wheels should pump liquid, and whose elasticity should be the means of storing electrostatic energy in the medium. It may be worth while pointing out some of the results of having shown that the energy of a disturbance of this medium would be represented by the same equations as those Maxwell has shown to hold for the æther. The most complicated results follow from supposing the faces of the cubes, of which the medium is constructed, to have different elasticities. Such a structure represents a crystalline medium. Its vibrations would be propagated according to the laws of propagation of light in crystalline media. The wave-surface would be Fresnel's wave-surface, and it would exhibit conical refraction. If the cubes were twisted, the structure would be like that of quartz or other substances that rotate the plane of polarization of light. In order to represent the rotation of the plane of polarization of light by magnetism, it would be necessary to introduce some mechanism connecting this æther with matter. That this would be required is evident from the consideration that no amount of inherent rotation of these wheels would alter the plane of polarization of a vibration transmitted by them. Now, although I am not prepared to suggest any actual method of connection, it may be worth while pointing out two different ways in which it may work.

In the first place the matter may be in rotation when subject to magnetic force ; and it may be connected with the æther in such a way that the direction of its axis and rotation is altered by the vibrations passing in the æther. It would then react on the æther in such a way as to rotate the plane of polarization of the wave. In the second place the matter might only act as a link connecting æther elements rotating in rectangular directions in such a way that the rotation of one element altered the axis of rotation of a rectangular element. The reaction of this latter on the former would then rotate the plane of polarization of a wave propagated parallel to the latter. The first of these connections is that in accordance with Maxwell's theory as to the connection between the æther and matter that he introduces in his 'Electricity and Magnetism,' in order to explain the rotation of the plane of polarization of light by magnetism ; while the second is the one that I supposed in my paper "On the Electromagnetic Theory of the Reflection and Refraction of Light" (R. S. Trans. 1880).

I need hardly say, in conclusion, that I do not in the least intend to convey the impression that the actual structure of the æther is a bit like what I have described. What physicists ought to look for is such a mode of motion in space as will confer upon it the properties required in order that it may exhibit electromagnetic phenomena. Such a mode of motion would be a real explanation of these phenomena. I have only given a description of them.

I think, however, that it is worth while considering these models, because in them the disturbance which represents light is not the same as the vibrations of an elastic jelly, for what represents an electric displacement is a change of structure of an element, and not a displacement of the element ; and it seems almost certain that, notwithstanding the very high authority which seems to support the view that the æther is *like* an elastic jelly, nevertheless its vibrations are much more of the nature of alterations in structure than of displacements.

XI. *On Binocular Glasses adjustable to Eyes having unequal Focal Lengths.* *By* Colonel MALCOLM, *R.E., C.B.*[*]

WHETHER many people are born with their eyes differing in focal length or not I cannot say; it is sufficient that I have arrived at a period of life when glasses are necessary to enable me to read with comfort, and also that I found a very highly treasured pair of binocular glasses, by Voigtländer, become yearly more difficult to use, and at last *useless*; but I found that I could see with equal distinctness with each eye, using only one eye at a time, on condition of a very slight alteration of focus.

Mr. Browning, optician, in the Strand, having tested my eyes for a pair of pince-nez, told me that, as regards the ordinary reading and writing distances at any rate, my eyes were not a pair, and that after a certain age few people's were who had used their eyes much.

Accepting the fact, I set to work to make my binocular glasses once more useful—in this way.

One tube is left untouched; the eyepiece of the other is so arranged that it can be moved through a small range in and out, with reference to the eyepiece of the untouched tube, by turning round a milled ring. An index arrangement is provided.

The unaltered tube is used with one eye and brought to the most perfect focus possible in the ordinary way; then the other tube is used with the other eye, and by means of the adjustment its definition is made as perfect as may be, the ordinary adjustment not being interfered with. The two eyes are then used together; and the process of adjustment had better be gone over again, as certainly the two eyes do help each other.

The final position of the index-mark is noted; and that holds good for all ranges, as far as I have tried.

Having noted this, you may lend your glasses to your friend, who may alter them to his sight, and yet have them in perfect order for yourself by bringing the index to your own mark.

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[*] Read March 28, 1885. The glasses were exhibited at the Meeting.

The increase of power in the glasses, and of comfort to the user of them, has been a surprise to almost all who have tried the improved adjustment.

I do not enlarge upon the way in which Mr. Browning has carried out the details; you can see for yourselves that the improvement in no way disfigures the glasses, and as the expense is 12*s.* 6*d.* for glasses made to the common pattern, I think that the improvement is within the reach of most people who use binoculars.

XII. *Summary of Lecture on Calculating Machines, delivered before the Physical Society of London, March* 28, 1885, *by* JOSEPH EDMONDSON, *of Halifax.*

CALCULATING machines are of two classes, the automatic and the semi-automatic. The former were invented by Mr. Charles Babbage between 1820 and 1834, and were designed mainly for the computation of tables. When the primary factors of a table are placed on such a machine, it will calculate the table and impress it line by line upon a stereo-mould with great celerity and absolute correctness. Mr. Babbage's machine, so far as it was completed, is in the South Kensington Museum; and another machine by M. Scheutz (for the principles of which he confessed his indebtedness to Mr. Babbage) is in the office of the Registrar General at Somerset House. These machines proceed by the method of Differences, and are therefore termed "Difference-Engines." Their work is limited to addition, but they indirectly perform subtraction by adding the complement of the subtrahend.

If it be required to construct a table (of cubes for instance), the law of its growth must be found by reckoning mentally the first few terms of the table. Placing them in a line, the difference between each term and its successor is placed under the former. This process is repeated with the differences until at last a line is reached in which all the differences are cyphers. Thus:—

Table . . .	A	1	8	27	64	125	216
1st difference	B	7	19	37	61	91	
2nd ,,	C	12	18	24	30		
3rd ,,	D	6	6	6			
4th ,,	E	0	0				

The machine may now be set to work by placing the first column of figures in their respective places upon it. During the first half of the first revolution of the machine, the wheels containing the terms A and C have the terms B and D added on to them and become 8 and 18 respectively. During the second half-revolution, B and D have the terms C and E added on to them, becoming 19 and 6 respectively, and the second column is complete. Each succeeding revolution of the machine similarly produces each succeeding column, the upper line being the term required for the table.

The apparatus consists of identical parts, equal in number to the digits required for the last column of the tables it is to compute. It is therefore very complex and very costly, and will not serve the purpose of computers in general, who must have recourse to the semi-automatic class of instruments. These are portable, of moderate cost, allow of very rapid working, and require no special mathematical skill.

In 1663 Sir Samuel Moreland produced an instrument by which additions and subtractions could be worked, digit by digit; but it took more time than the ordinary mental operation. It was left to Viscount Mahon (afterwards Earl of Stanhope) to produce the first really practical instrument. Besides a machine dated 1780, which was a great advance on that of Sir Samuel Moreland, though on the same lines, he invented three machines. Those of 1775 and 1777 were on the table for inspection after the lecture, and abound in beautiful and effective contrivances. The third machine the lecturer had not seen, and it has never been described. In that of 1775 is found the "Stepped Reckoner," the basis of the only instruments that have come into extensive use.

The reckoner of the modern machines, patented in 1851 by M. Thomas de Colmar, consists of a cylinder divided into 10 sections, on which there are respectively 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 teeth. The teeth of one section being coincident with an equal number of those of the next section, the whole presents a stepped appearance. Each revolution of this reckoner moves a pinion of 10 teeth, in gear with one of its sections, as many teeth as there are upon that section. The motion of the pinion is communicated to a dial with the digits 0 to 9 in orderly succession upon it. Thus, if the figure seen through an aperture in the covering of the dial were 0, and the pinion were in gear with the section having 3 teeth, the first revolution of the reckoner would move the dial to 3 ; the second to 6 ; the third to 9 ; this result being the multiple of 3 (teeth on the reckoner) by 3 (revolutions). A series of, say, 8 reckoners, pinions and dials, each pinion being set to the section having 3 teeth, would give in 3 revolutions the product 99,999,999. The next revolution would bring in a new feature—the carrying of the tens. Here lies the great difficulty of Calculating Machines ; but the difficulty has been overcome, though not without leaving room for improvement. The 1 that will have to be carried from each dial to that immediately on its left cannot be added while the latter is being operated upon by its reckoner. The machine must therefore make an independent note of it until it can be dealt with. Each dial has a tooth (the *primary* carrying-tooth) which makes this note at the moment that the dial passes from 9 to 0, by pushing along the axis of the reckoner belonging to the next higher digit a sliding-piece which revolves with the reckoner and carries a single tooth (the *secondary* carrying-tooth). Above this sliding-piece, and upon the same axis as the pinion in gear with the reckoner, is a second pinion of 10 teeth. Before the sliding-piece was moved by the primary carrying-tooth, the secondary carrying-tooth passed on the side of this second pinion without affecting it ; but being now brought into line with it, the pinion will be moved 1 tooth, and will therefore add 1, as soon as the reckoner has added its own proper number ; so that the fourth revolution, giving the product of 33,333,333 by 4, brings the dials to 133,333 332. When the secondary tooth has done its work, an incline on its side comes in contact with a projection on the frame of the machine, which restores it to its original inoperative position. Each reckoner is set to operate a little later than its neighbour to the right, so that the carrying on the latter may be completed before that on the former is commenced. The before-named blank space on the reckoner is to allow time for the carrying ; and for a "dead-point" in the machine, at which the first-named pinions, which are movable longitudinally on their axes, may be set to their required sections of the reckoners, and other changes may be made in the setting of the machine. The position of the pinions is shown by indicators on the face of the machine, each of which can be set to any required figure.

Each pinion communicates its motion to its dial by bevel-gear in the well-known combination of three wheels, which permit the action to be reversed at will, so that the change from + to − or from × to ÷ is made instantaneously.

In multiplying by more than a single digit, the dials require to change their position in relation to the multiplicand, to imitate the "stepping" of the lines in long multiplication. They are therefore placed in a slide which can be moved, step by step, from right to left or *vice versâ.*

Up to a recent date the machine of M. Thomas de Colmar was the only one in use. A few years ago Mr. George B. Grant, of Boston, Mass., brought out an instrument somewhat on the lines of the Stanhope machine of 1777. It is very compact, is beautifully and substantially made; but as subtraction is performed by adding the complement of the subtrahend, reversing is so tedious as to be fatal to the general adoption of the machine. Its range is also very limited, and there are other defects inherent in the method of its construction.

Still more recently Mr. Tate, of London, has introduced a machine on the lines of M. Thomas, in the internal mechanism of which there are several improvements. The workmanship, too, is very superior to that of the French instrument.

Mr. Edmondson (the lecturer) having, in practice, found certain inconvenient limits to the utility of these machines, conceived the idea of obviating them by throwing the instrument into the circular form, and thus obtaining an *endless* instead of a limited slide. This has produced a machine of greatly extended powers; for it can deal with a multiplier or dividend of any number of figures, and can carry a quotient to any number of decimal places. But its chief utility does not consist in dealing with such large numbers. It frequently happens that quotients produced on a machine require to be further operated upon; but as in the straight machines they are recorded on a special set of dials, which are not in the general working line, they must be transferred by hand before they can be dealt with in any other way than as multipliers.

In the circular machine they come up in the working line ready for further operations. The circular form, in fact, gives a certain elasticity in the applications of the machine which could not be described within the limits of the lecture, and which can only be appreciated by actual use of the instrument. Although the special set of dials for recording multipliers and quotients are dispensed with as non-essential in the circular machine, they could be easily and conveniently added, and would still further extend the powers of the instrument.

The lecturer expressed his indebtedness to the following gentlemen for the loan of valuable instruments on the table, which, along with Mr. Tate's machine and several belonging to the lecturer himself, attracted much attention after the lecture :—

General Babbage, Bromley, Kent. A piece of the Difference-Engine of the late Charles Babbage, Esq. ; Stanhope Machines of 1775 and 1777 ; Sir Samuel Moreland's Machine of 1663.

The Rev. Professor Harley, F.R.S., Huddersfield. Stanhope Machine of 1780.

Theodore B. Jones, Esq., Harrogate. The American Machine of George B. Grant, Esq.

F. H. Bowman, Esq., D.Sc., Halifax. A Machine of Sir Samuel Moreland's, constructed on the principle of "Napier's Bones."

XIII. *On a Chrono-barometer and Chrono-thermometer.*

By W. Ford Stanley*.

These instruments consisted of clocks regulated by pendulums, formed, in the first instrument, of a mercurial barometer, and in the second of a similar barometer inclosed in a hermetically-sealed air-chamber, the inclosed barometer thus acting as an air-thermometer. Increase of pressure in the one case, and of temperature in the other, causes the mercury to rise, and thus accelerates the pendulum. By the gain or loss of time the mean pressure or temperature can be calculated for any period.

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* Read March 28, 1885.